CHAPTER TEN

10.1 Derivability

1. a. Derive: $(\forall y)Fy$

1	$(\forall x)Fx$	Assumption
2	Fa	1 $\forall$E
3	$(\forall y)Fy$	2 $\forall$I

c. Derive: $(\exists x)(\exists y)Hxy$

1	$(\forall x)(\forall y)Hxy$	Assumption
2	$(\forall y)Hay$	1 $\forall$E
3	Hab	2 $\forall$E
4	$(\exists y)Hay$	3 $\exists$I
5	$(\exists x)(\exists y)Hxy$	4 $\exists$I

e. Derive: Kg

1	$(\forall x)(\forall y)Hxy$	Assumption
2	$Hab \supset Kg$	Assumption
3	$(\forall y)Hay$	1 $\forall$E
4	Hab	3 $\forall$E
5	Kg	2, 4 $\supset$E

g. Derive: $(\exists y)Wy$

1	$(\forall x)Sx$	Assumption
2	$(\exists y)Sy \supset (\forall w)Ww$	Assumption
3	Sa	1 $\forall$E
4	$(\exists y)Sy$	3 $\exists$I
5	$(\forall w)Ww$	2, 4 $\supset$E
6	Wa	5 $\forall$E
7	$(\exists y)Wy$	6 $\exists$I

i. Derive: $(\exists x)(Lxx \,\&\, Hxx)$

1	$(\forall x)(\forall y)Lxy$	Assumption
2	$(\exists w)Hww$	Assumption
3		Haa	A / $\exists$E
4		$(\forall y)Lay$	1 $\forall$E
5		Laa	4 $\forall$E
6		$Laa \,\&\, Haa$	3, 6 &I
7		$(\exists x)(Lxx \,\&\, Hxx)$	6 $\exists$I
8	$(\exists x)(Lxx \,\&\, Hxx)$	2, 3–7 $\exists$E

2. The mistakes in the attempted derivations are indicated and explained below.

a. Derive: Na

1	$(\forall x)Hx \supset \sim (\exists y)Ky$	Assumption	
2	$Ha \supset Na$	Assumption	
3	Ha	1 $\forall$E	**MISTAKE!**
4	Na	2, 3 $\supset$E	

Universal Elimination is a rule of inference. Like all rules of inference, it can be applied only to whole sentences, not to a formula or sentence that is a component of a larger sentence, and '$(\forall x)Hx$' is a component of the larger sentence, namely '$(\forall x)Hx \supset \sim (\exists y)Ky$.

c. Derive: $(\exists x)Cx$

1	$(\exists y)Fy$	Assumption	
2	$(\forall w)(Fw \equiv Cw)$	Assumption	
3	Fa	1 $\exists$E	**MISTAKE!**
4	$Fa \equiv Ca$	2 $\forall$E	
5	Ca	3, 4 $\equiv$E	
6	$(\exists x)Cx$	5 $\exists$I	

Existential Elimination is a rule that requires the construction of a subderivation. Here is a correctly done derivation:

Derive: $(\exists x)Cx$

1	$(\exists y)Fy$	Assumption
2	$(\forall w)(Fw \equiv Cw)$	Assumption
3	Fa	1 / $\exists$E
4	$Fa \equiv Ca$	2 $\forall$E
5	Ca	3, 4 $\equiv$E
6	$(\exists x)Cx$	5 $\exists$I
7	$(\exists x)Cx$	2, 3–6 $\exists$E

e. Derive: $(\exists y)(\forall x)Ayx$

1	$(\forall x)(\exists y)Ayx$	Assumption	
2	$(\forall x)Aax$	1 $\forall$E	**MISTAKE!**
3	$(\exists y)(\forall x)Ayx$	2 $\exists$I	

Universal Elimination takes us from a Universally quantified sentence to a substitution instance of that sentence. Here we start with a universally quantified sentence but instead of dropping the universal quantifier the existential quantifier, which comes after the universal quantifier, has been dropped. There is no correct derivation in this case. The sentence on line 3 is not derivable in *PD* from the sentence on line 1.

10.2E EXERCISE ANSWERS

1. Validity

a. Derive: (∀x)(Fx ⊃ Hx)

1	(∀y)[Fy ⊃ (Gy & Hy)]	Assumption
2	Fc	A / ⊃I
3	Fc ⊃ (Gc & Hc)	1 ∀E
4	Gc & Hc	2, 3 ⊃E
5	Hc	4 &E
6	Fc ⊃ Hc	2–5 ⊃I
7	(∀x)(Fx ⊃ Hx)	6 ∀I

#c. Our derivation of the conclusion from the premises will use Universal Elimination, Existential Elimination, and Existential Introduction. We will make Existential Elimination our primary strategy:

Derive: (∃z)Fz

1	(∀y)[Gy ⊃ (Hy & Fy)]	Assumption
2	(∃x)Gx	Assumption
3	Ga	A / ∃E
G	(∃z)Fz	
G	(∃z)Fz	2, 3–__ ∃E

We will next use Universal Elimination to obtain a material conditional whose antecedent is 'Ga', allowing us to use Conditional Elimination to obtain 'Ha & Fa'. The rest is straightforward:

Derive: (∃z)Fz

1	(∀y)[Gy ⊃ (Hy & Fy)]	Assumption
2	(∃x)Gx	Assumption
3	Ga	A / ∃E
4	Ga ⊃ (Ha & Fa)	1 ∀E
5	Ha & Fa	3, 4 ⊃E
6	Fa	5 &E
7	(∃z)Fz	6 ∃I
8	(∃z)Fz	2, 3–7 ∃E

e. Derive: $(\forall x)Hx$

1	$(\exists x)Fx \supset (\forall x)Gx$	Assumption
2	Fa	Assumption
3	$(\forall x)(Gx \supset Hx)$	Assumption
4	$(\exists x)Fx$	2 ∃I
5	$(\forall x)Gx$	1, 4 ⊃E
6	Gb	5 ∀E
7	$Gb \supset Hb$	3 ∀E
8	Hb	6, 7 ⊃E
9	$(\forall x)Hx$	8 ∀I

Note that it is essential that the constant chosen as the instantiating constant in line 6 be other than ‘a’, for ‘a’ occurs in an open assumption and were ‘a’ also used at line 6 we would violate the first restriction on Universal Introduction at line 9—for the instantiating constant, ‘a’, would then occur in an open assumption (on line 2).

g. Derive: $(\forall x)(Fx \vee Gx)$

1	$(\forall x)Fx \vee (\forall x)Gx$	Assumption
2	$\quad (\forall x)Fx$	A / ∨E
3	$\quad Fa$	2 ∀E
4	$\quad Fa \vee Ga$	3 ∨I
5	$\quad (\forall x)Gx$	A / ∨E
6	$\quad Ga$	5 ∨E
7	$\quad Fa \vee Ga$	6 ∨I
8	$Fa \vee Ga$	1, 2–4, 5–7 ∨E
9	$(\forall x)(Fx \vee Gx)$	8 ∀I

#i. Since the conclusion is a universally quantified sentence and there are no existentially quantified sentences among the premises, we will plan on deriving the conclusion by Universal Introduction and use Conditional Introduction to derive the substitution instance to which we will apply Universal Introduction:

Derive: $(\forall y)[(Fy \vee Gy) \supset Hy]$

1	$(\forall x)(Fx \supset Hx)$	Assumption
2	$(\forall y)(Gy \supset Hy)$	Assumption
3	$\quad Fb \vee Gb$	A / ⊃I
G	$\quad Hb$	
G	$(Fb \vee Gb) \supset Hb$	3–__ ⊃I
G	$(\forall y)[(Fy \vee Gy) \supset Hy]$	__ ∀I

Our plan will not violate the second restriction on Universal Introduction, for while the instantiating constant ‘b’ does occur in an assumption (at line 3), that assumpion will be closed at the point where we use Universal Introduction (the last line). The assumption on line 3 is a disjunction and we will now use Disjunction Elimination to obtain ‘Hb’. To do so we will have to use Universal Elimination twice, once in association with each subderivation of the Disjunction Elimination strategy:

Derive: (∀y)[(Fy ∨ Gy) ⊃ Hy]

1	(∀x)(Fx ⊃ Hx)	Assumption
2	(∀y)(Gy ⊃ Hx)	Assumption
3	Fa ∨ Ga	A/ ⊃I
4	Fa	A / ∨E
5	Fa ⊃ Ha	1 ∀E
6	Ha	4, 5 ⊃E
7	Ga	A / ∨E
8	Ga ⊃ Ha	2 ∀E
9	Ha	7, 8 ⊃E
10	Ha	3, 4–6, 7–9 ∨E
11	(Fa ∨ Ga) ⊃ Ha	3–10 ⊃I
12	(∀y)[(Fy ∨ Gy) ⊃ Hy]	11 ∀I

k. Derive: (∀x)(Fx ⊃ Gx)

1	(∃x)Hx	Assumption
2.	(∀x)(Hx ⊃ Rx)	Assumption
3.	(∃x)Rx ⊃ (∀x)Gx	Assumption
4	Ha	A / ∃E
5	Ha ⊃ Ra	2 ∀E
6	Ra	4, 5 ⊃E
7	(∃x)Rx	6 ∃I
8	(∀x)Gx	3, 7 ⊃E
9	Fb	A / ⊃I
10	Gb	8 ∀E
11	Fb ⊃ Gb	9–10 ⊃I
12	(∀x)(Fx ⊃ Gx)	11 ∀I
13	(∀x)(Fx ⊃ Gx)	3, 4–12 ∃E

m. Derive: (∃y)(Hy ∨ Jy)

1	(∀x)Fx ∨ (∀y) ~ Gy	Assumption
2	Fa ⊃ Hb	Assumption
3	~ Gb ⊃ Jb	Assumption
4	(∀x)Fx	A / ∨E
5	Fa	4 ∀E
6	Hb	2, 5 ⊃E
7	Hb ∨ Jb	6 ∨I
8	(∃y)(Hy ∨ Jy)	7 ∃I
9	(∀y) ~ Gy	A / ∨E
10	~ Gb	9 ∀E
11	Jb	3, 10 ⊃E
12	Hb ∨ Jb	11 ∨I
13	(∃y)(Hy ∨ Jy)	12 ∃I
14	(∃y)(Hy ∨ Jy)	1, 4–8, 9–13 ∨E

2. Theorems

a. Derive: Fa ⊃ (∃y)Fy

1	Fa	A / ⊃I
2	(∃y)Fy	1 ∃I
3.	Fa ⊃ (∃y)Fy	1–2 ⊃I

c. Derive: (∀x)[Fx ⊃ (Gx ⊃ Fx)]

1	Fa	A / ⊃I
2	Ga	A / ⊃I
3	Fa	1 R
4	Ga ⊃ Fa	2–3 ⊃I
5	Fa ⊃ (Ga ⊃ Fa)	1–4 ⊃I
6	(∀x)[Fx ⊃ (Gx ⊃ Fx)]	5 ∀I

e. Derive: ~ (∃x)Fx ⊃ (∀x) ~ Fx

1	~ (∃x)Fa	A / ⊃I
2	Fa	A / ~ I
3	(∃x)Fx	2 ∃I
4	~ (∃x)Fx	1 R
5	~ Fa	2–4 ~ I
6	(∀x) ~ Fx	5 ∀I
7	~(∃x)Fx ⊃ (∀x) ~ Fx	1–6 ⊃I

g. Derive: $Fa \vee (\exists y) \sim Fy$

1	$\sim (Fa \vee (\exists y) \sim Fy)$	A / ~ E
2	Fa	A / ~ I
3	$Fa \vee (\exists y) \sim Fy$	2 ∨I
4	$\sim (Fa \vee (\exists y) \sim Fy$	1 R
5	$\sim Fa$	2–4 ~ I
6	$(\exists y) \sim Fy$	5 ∃I
7	$Fa \vee (\exists y) \sim Fy$	6 ∨I
8	$\sim (Fa \vee (\exists y) \sim Fy)$	1 R
9	$Fa \vee (\exists y) \sim Fy$	1–8 ~ E

#i. Since the theorem we want to prove is a material conditional, our primary strategy will be Conditional Introduction.

Derive: $[(\forall x)Fx \vee (\forall x)Gx] \supset (\forall x)(Fx \vee Gx)]$

1	$(\forall x)Fx \vee (\forall x)Gx$	A / ⊃I
G	$(\forall x)(Fx \vee Gx)$	
G	$[(\forall x)Fx \vee (\forall x)Gx] \supset (\forall x)(Fx \vee Gx)]$	1–__ ⊃I

Our only accessible assumption is a disjunction, and our current goal is a universally quantified sentence. This suggests we will be using both Disjunction Elimination and Universal Introduction. The question is whether the goal of our Disjunction Elimination strategy should be '$(\forall x)(Fx \vee Gx)$' or a substitution instance of that sentence, say '$Fb \vee Gb$', with the intent of using Universal Introduction after we have used Disjunction Elimination. It turns out that both approaches will work. We will use the latter approach:

Derive: [(∀x)Fx ∨ (∀x)Gx] ⊃ (∀x)(Fx ∨ Gx)]

1	(∀x)Fx ∨ (∀x)Gx	A / ⊃I
2	(∀x)Fx	A / ∨E
G	Fb ∨ Gb	
	(∀x)Gx	A / ∨E
G	Fb ∨ Gb	
G	Fb ∨ Gb	1, 2–__, __–__ ∨E
G	(∀x)(Fx ∨ Gx)	__ ∀I
G	[(∀x)Fx ∨ (∀x)Gx] ⊃ (∀x)(Fx ∨ Gx)]	1–__ ⊃I

Completing the two Disjunction Elimination subderivations is straightforward. In each case we will use Universal Elimination followed by Disjunction Introduction. To make this work we must, of course, in both cases use 'b' as our instantiating constant:

Derive: [(∀x)Fx ∨ (∀x)Gx] ⊃ (∀x)(Fx ∨ Gx)

1	(∀x)Fx ∨ (∀x)Gx	A / ⊃I
2	(∀x)Fx	A / ∨E
3	Fb	2 ∀E
4	Fb ∨ Gb	3 ∨I
5	(∀x)Gx	A / ∨E
6	Gb	5 ∀E
7	Fb ∨ Gb	6 ∨I
8	Fb ∨ Gb	1, 2–4, 5–7 ∨E
9	(∀x)(Fx ∨ Gx)	8 ∀I
10	[(∀x)Fx ∨ (∀x)Gx] ⊃ (∀x)(Fx ∨ Gx)	1–9 ⊃I

Note that we could have done Universal Introduction within each of our innermost subderivations, thereby obtaining '(∀x)(Fx ∨ Gx)' rather than 'Fb ∨ Gb' by Disjunction Elimination. Doing so would produce a derivation that is one line longer.

k. Derive: (∃x)(Fx & Gx) ⊃ [(∃x)Fx & (∃x)Gx]

1	(∃x)(Fx & Gx)	A / ⊃I
2	Fa & Ga	A / ∃E
3	Fa	2 &E
4	(∃x)Fx	3 ∃I
5	Ga	2 &E
6	(∃x)Gx	5 ∃I
7	(∃x)Fx & (∃x)Gx	4, 6 &I
8	(∃x)Fx & (∃x)Gx	1, 2–7 ∃E
9	(∃x)(Fx & Gx) ⊃ [(∃x)Fx & (∃x)Gx]	1–8 ⊃I

m. Derive: (∀x)Hx ≡ ~ (∃x) ~ Hx

1	│ │ (∀x)Hx	A / ≡I
2	│ │ │ (∃x) ~ Hx	A / ~ I
3	│ │ │ │ ~ Ha	A / ∃E
4	│ │ │ │ │ (∀x)Hx	A / ~ I
5	│ │ │ │ │ ~ Ha	3 R
6	│ │ │ │ │ Ha	1 ∀E
7	│ │ │ │ ~ (∀x)Hx	4–6 ~ I
8	│ │ │ ~ (∀x)Hx	2, 3–7 ∃E
9	│ │ │ (∀x)Hx	1 R
10	│ │ ~ (∃x) ~ Hx	2–9 ~ I
11	│ │ ~ (∃x) ~ Hx	A / ≡I
12	│ │ │ ~ Hb	A / ~ E
13	│ │ │ ~ (∃x) ~ Hx	11 R
14	│ │ │ (∃x) ~ Hx	12 ∃I
15	│ │ Hb	12–14 ~ E
16	│ │ (∀x)Hx	15 ∀I
17	│ (∀x)Hx ≡ ~ (∃x) ~ Hx	1–10, 11–16 ≡I

3. Equivalence

a. Derive: (∀x)Fx & (∀x)Gx

1	│ (∀x)(Fx & Gx)	Assumption
2	│ Fa & Ga	1 ∀E
3	│ Fa	2 &E
4	│ (∀x)Fx	3 ∀I
5	│ Ga	2 &E
6	│ (∀x)Gx	5 ∀I
7	│ (∀x)Fx & (∀x)Gx	4, 6 &I

Derive: (∀x)(Fx & Gx)

1	│ (∀x)Fx & (∀x)Gx	Assumption
2	│ (∀x)Fx	1 &E
3	│ Fa	2 ∀E
4	│ (∀x)Gx	1 &E
5	│ Ga	4 ∀E
6	│ Fa & Ga	3, 5 &I
7	│ (∀x)(Fx & Gx)	6 ∀I

c. Derive: ~ (∃x) ~ Fx

1	(∀x)Fx	Assumption
2	(∃x) ~ Fx	A / ~I
3	~ Fa	A / ∃E
4	(∀x)Fx	A / ~ I
5	Fa	4 ∀E
6	~ Fa	3 R
7	~ (∀x)Fx	4–6 ~ I
9	~ (∀x)Fx	2, 3–7 ∃E
10	(∀x)Fx	1 R
11	~ (∃x) ~ Fx	2–10 ~ I

Derive: (∀x)Fx

1	~ (∃x) ~ Fx	Assumption
2	~ Fa	A / ~ E
3	(∃x) ~ Fx	2 ∃I
4	~ (∃x) ~ Fx	1 R
5	Fa	2–4 ~ E
6	(∀x)Fx	5 ∀I

#e. Derive: ~ (∀x) ~ Fx

1	(∃x)Fx	Assumption
G	~ (∀x) ~ Fx	

The one primary assumption of our derivation is an existentially quantified sentence, suggesting Existential Elimination as a possible strategy. The goal sentence is a negation, suggesting Negation Introduction. In fact, we will use both strategies, one within the other. In our first attempt we will use Existential Elimination as our primary strategy:

Derive: ~ (∀x) ~ Fx

1	(∃x)Fx	Assumption
2	Fa	A / ∃E
3	(∀x) ~ Fx	A / ~ I
G	~ (∀x) ~ Fx	3–__ ~I
G	~ (∀x) ~ Fx	1, 2–__ ∃E

We have taken '~ (∀x) ~ Fx' as our goal, within our Existential Elimination subderivation. Note that this sentence does not contain the constant 'a', so we are in no danger of violating the third restriction on Existential Elimination (that the instantiating constant not occur in the derived sentence). To complete the derivation we need to derive a sentence and its negation within the scope of the assumption on line 3. Only one negation is readily available, '~ Fa', which can be obtained by applying Universal Elimination to '(∀x) ~ Fx' on line 3. And 'Fa' can be obtained by Reiteration. So the completed derivation is

Derive: ~ (∀x) ~ Fx

1	(∃x)Fx	Assumption
2	Fa	A / ∃E
3	(∀x) ~ Fx	A / ~ I
4	~ Fa	3 ∀E
5	Fa	2 R
6	~ (∀x) ~ Fx	3–5 ~ I
7	~ (∀x) ~ Fx	1, 2–6 ∃E

To avoid violating the third restriction on Existential Elimination it is a good idea, at the time an Existential Elimination subderivation is started, to select the goal of that subderivation; making sure that the goal sentence does not contain the instantiating constant in the subderivation's assumption. In a derivation that uses Existential Elimination as its primary strategy the sentence that occurs on the last line should also appear as the last sentence in the subderivation. In this example that sentence is '~ (∀x) ~ Fx'.

To complete our demonstration that '(∃x)Fx' and '~ (∀x) ~ Fx' are equivalent we will now derive the first sentence from the second:

Derive: (∃x)Fx

1	~ (∀x) ~ Fx	Assumption
G	(∃x)Fx	

Here our goal sentence is an existentially quantified sentence, and our one primary assumption a negation. The former suggests Existential Introduction as a strategy, the latter suggests Negation Elimination (since we do have a negation readily available). We will construct two derivations to illustrate that both strategies work as the primary strategy, in each case sing the order strategy as a secondary strategy:

Derive: $(\exists x)Fx$

1	$\sim (\forall x) \sim Fx$	Assumption
2	$\sim (\exists x)Fx$	A / ~ E
G	$(\forall x) \sim Fx$	
	$\sim (\forall x) \sim Fx$	1 R
G	$(\exists x)Fx$	2–__ ~ E

We have decided to use '$(\forall x) \sim Fx$' and '$\sim (\forall x) \sim Fx$' as the sentence and negation we derive for Negation Elimination. (We could of course, also have decided to use '$(\exists x)Fx$' and '$\sim (\exists x)Fx$'.) Our current goal is '$(\forall x) \sim Fx$', a universally quantified sentence. One way to obtain it is by Universal Introduction, which will require obtaining a substitution instance of that sentence. In planning for Universal Introduction we pick as our goal a substitution instance of the desired universally quantified sentence, and the instantiating constant in this substitution instance should not occur in any open assumption. Because neither of our assumptions contains a constant, we are free to choose any constant. We choose the substitution instance '~ Fa'. And since this sentence is a negation, we will try to obtain it by Negation Introduction:

Derive: $(\exists x)Fx$

1	$\sim (\forall x) \sim Fx$	Assumption
2	$\sim (\exists x)Fx$	A / ~ E
3	Fa	A / ~ I
G	$\sim Fa$	3–__ ~ I
G	$(\forall x) \sim Fx$	__ ∀I
	$\sim (\forall x) \sim Fx$	1 R
G	$(\exists x)Fx$	2–__ ~ E

As of line 3 two negations are available to us, '$\sim (\forall x) \sim Fx$' and '$\sim (\exists x)Fx$'. We select the latter to use within the negation strategy that begins at line 3 because the unnegated '(∃x)Fx' is easily obtainable from line 3 by Existential Introduction:

Derive: (∃x)Fx

1	~ (∀x) ~ Fx	Assumption
2	~ (∃x)Fx	A / ~E
3	Fa	A / ~ I
4	(∃x)Fx	3 ∃I
5	~ (∃x)Fx	2 R
6	~ Fa	3–5 ~ I
7	(∀x) ~ Fx	6 ∀I
8	~ (∀x) ~ Fx	1 R
9	(∃x)Fx	2–8 ~ E

We have now derived each member of our original pair of sentences from the other, so we have demonstrated that these sentences, '(∃x)Fx' and '~ (∀x) ~ Fx' are equivalent in *PD*.

g. Derive: ~ (∃y)(Hy & Iy)

1	(∀z)(Hz ⊃ ~ Iz)	Assumption
2	(∃y)(Hy & Iy)	A / ~ I
3	Hb & Ib	A / ∃E
4	(∀z)(Hz ⊃ ~ Iz)	A / ~ I
5	Hb ⊃ ~ Ib	1 ∀E
6	Hb	3 &E
7	~ Ib	5, 6 ⊃E
8	Ib	3 &E
9	~ (∀z)(Hz ⊃ ~ Iz)	4–8 ~ I
10	~ (∀z)(Hz ⊃ ~ Iz)	2, 3–9 ∃E
11	(∀z)(Hz ⊃ ~ Iz)	1 R
12	~ (∃y)(Hy & Iy)	2–11 ~ I

Derive: (∀z)(Hz ⊃ ~ Iz)

1	~ (∃y)(Hy & Iy)	Assumption
2	Ha	A / ⊃I
3	Ia	A / ~ I
4	Ha & Ia	2, 3 &I
5	(∃y)(Hy & Iy)	4 ∃I
6	~ (∃y)(Hy & Iy)	1 R
7	~ Ia	3–6 ⊃I
8	Ha ⊃ ~ Ia	2–7 ⊃I
9	(∀z)(Hz ⊃ ~ Iz)	8 ∀I

i. Derive: (∀x) (Fx ⊃ (∃y) Gy)

1	(∀x) (∃y) (Fx ⊃ Gy)	Assumption
2	(∃y) (Fa ⊃ Gy)	1 ∀E
3	Fa ⊃ Gb	A / ∃E
4	Fa	A / ⊃I
5	Gb	3, 4 ⊃I
6	(∃y) Gy	5 ∃I
7	Fa ⊃ (∃y) Gy	4–6 ⊃I
8	Fa ⊃ (∃y) Gy	2, 3–7 ∃E
9	(∀x) (Fx ⊃ (∃y) Gy)	8 ∀I

Derive: (∀x) (∃y) (Fx ⊃ Gy)

1	(∀x) (Fx ⊃ (∃y) Gy)	Assumption
2	~ (∃y) (Fa ⊃ Gy)	A / ~ E
3	Fa	A / ⊃I
5	Fa ⊃ (∃y) Gy	1 ∀E
6	(∃y) Gy	3, 5 ⊃E
7	Gc	A / ∃E
8	~ Gb	A / ~ E
9	Fa	A / ⊃I
10	Gc	7 R
11	Fa ⊃ Gc	9–10 ⊃I
12	(∃y) (Fa ⊃ Gy)	11 ∃I
13	~ (∃y) (Fa ⊃ Gy)	2 R
14	Gb	8–13 ~ E
15	Gb	6, 7–14 ∃E
16	Fa ⊃ Gb	3–15 ⊃I
17	(∃y) (Fa ⊃ Gy)	16 ∃I
18	~ (∃y) (Fa ⊃ Gy)	2 R
19	(∃y) (Fa ⊃ Gy)	2–18 ∃E
20	(∀x) (∃y) (Fx ⊃ Gy)	19 ∀I

4. Inconsistency

a. Derive: Fa, ~ Fa

1	(∀x) (Fx ≡ ~ Fx)	Assumption
2	Fa ≡ ~ Fa	1 ∀E
3	Fa	A / ~ I
4	~ Fa	2, 3 ≡E
5	Fa	3 R
6	~ Fa	3–5 ~ I
7	Fa	2, 6 ≡E

#c. It is fairly easy to see that the set {~ (∀x)Fx, ~ (∃x) ~ Fx} is inconsistent. If not everything is F, then there must be something that is not F, but this contradicts the claim that there is not something that is not F. The set contains two negations. We choose to use one of them, '~ (∀x)Fx', as ~ Q. Our derivation starts thus:

Derive: (∀x)Fx, ~ (∀x)Fx

1	~ (∀x)Fx	Assumption
2	~ (∃x) ~ Fx	Assumption
G	(∀x)Fx	
	~ (∀x)Fx	1 R

How we should continue is not immediately clear. We reason as follows: The sentences that are accessible include only two negations. There is no rule of inference that can be applied to a negation to yield a further sentence (Negation Elimination starts with the auxiliary assumption of a negation, not with a primary assumption that is a negation.) So working from the "top down" is not here promising. Our current goal is a universally quantified sentence, and Universal Introduction is the rule that yields such sentences. So we will plan on using Universal Introduction. To use it, we must first derive a substitution instance of our goal sentence. Since there are no constants in the primary assumptions, which substitution instance doesn't matter. We pick 'Fa'.

Derive: (∀x)Fx, ~ (∀x)Fx

1	~ (∀x)Fx	Assumption
2	~ (∃x) ~ Fx	Assumption
G	Fa	
G	(∀x)Fx	__ ∀I
	~ (∀x)Fx	1 R

The task now is to derive 'Fa'. We have added to new assumptions, so working from the "top down" is still not promising. So we will try to get 'Fa' by Negation Elimination:

Derive: (∀x)Fx, ~ (∀x)Fx

1	~ (∀x)Fx	Assumption
2	~ (∃x) ~ Fx	Assumption
3	~ Fa	A / ~ E
G	Fa	
G	(∀x)Fx	___ ∀I
	~ (∀x)Fx	1 R

With our new assumption, we can now work from the "top down". More specifically, we have '~ (∃x) ~ Fx' at line 2 and from line 3 we can obtain, by Existential Introduction, '(∃x) ~ Fx', giving us the **Q** and ~ **Q** we need to complete our Negation Elimination strategy and the derivation:

Derive: (∀x)Fx, ~ (∀x)Fx

1	~ (∀x)Fx	Assumption
2	~ (∃x) ~ Fx	Assumption
3	~ Fa	A / ~ E
4	(∃x) ~ Fx	3 ∃I
5	~ (∃x) ~ Fx	2 R
6	Fa	3–5 ~ E
7	(∀x)Fx	6 ∀I
8	~ (∀x)Fx	1 R

Our demonstration of inconsistency in PD is now complete. We have used Universal Introduction and met both restrictions on that rule: the instantiating constant 'a' does not occur in the sentence derived by Universal Introduction and it does not occur, as of line 7, in any open assumption.

e. Derive: $(\exists x)Gx$, $\sim(\exists x)Gx$

1	$(\forall x)(Fx \supset Gx)$	Assumption
2	$(\exists x)Fx$	Assumption
3	$\sim(\exists x)Gx$	Assumption
4	$\quad Fb$	A / ∃E
5	$\quad Fb \supset Gb$	1 ∀E
6	$\quad Gb$	4, 5 ⊃E
7	$\quad (\exists x)Gx$	6 ∃I
8	$(\exists x)Gx$	2, 4–7 ∃E
9	$\sim(\exists x)Gx$	3 R

g. Derive: $(\forall x)Fx$, $\sim(\forall x)Fx$

1	$(\forall x)Fx$	Assumption
2	$(\exists y)\sim Fy$	Assumption
3	$\quad \sim Fa$	A / ∃E
4	$\quad\quad (\forall x)Fx$	A / ~ I
5	$\quad\quad Fa$	1 ∀E
6	$\quad\quad \sim Fa$	3 R
7	$\quad \sim(\forall x)Fx$	4–6 ~ I
8	$\sim(\forall x)Fx$	2, 3–7 ∃E
9	$(\forall x)Fx$	1 R

i. Derive: $(\forall x)Fx$, $\sim(\forall x)Fx$

1	$(\forall x)(Hx \equiv \sim Gx)$	Assumption
2	$(\exists x)Hx$	Assumption
3	$(\forall x)Gx$	Assumption
4	$\quad Hc$	A / ∃E
5	$\quad\quad (\forall x)Gx$	A / ~ I
6	$\quad\quad Hc \equiv \sim Gc$	1 ∀E
7	$\quad\quad \sim Gc$	4, 6 ≡E
8	$\quad\quad Gc$	3 ∀E
9	$\quad \sim(\forall x)Gx$	5–8 ~ I
10	$\sim(\forall x)Gx$	2, 4–9 ∃E
11	$(\forall x)Gx$	3 R

k. Derive: (∃y)(Ry & My), ~ (∃y)(Ry & My)

1	(∀z)[Rz ⊃ (Tz & ~ Mz)]	Assumption
2	(∃y)(Ry & My)	Assumption
3	Ra & Ma	A / ∃E
4	(∃y)(Ry & My)	A / ~ I
5	Ra ⊃ (Ta & ~ Ma)	1 ∀E
6	Ra	3 &E
7	Ta & ~ Ma	5, 6 ⊃E
8	~ Ma	7 &E
9	Ma	3 &E
10	~ (∃y)(Ry & My)	4–9 ~ I
11	~ (∃y)(Ry & My)	2, 3–10 ∃E
12	(∃y)(Ry & My)	2 R

5. Derivability

a. Derive: (∀x)(∃y)Fxy

1	(∃y)(∀x)Fxy	Assumption
2	(∀x)Fxa	A / ∃E
3	Fba	2 ∀E
4	(∃y)Fby	3 ∃I
5	(∃y)Fby	1, 2–3 ∃E
6	(∀x)(∃y)Fxy	5 ∀I

c. Derive: (∃x)(∃y)(∃z)Fxyz

1	(∃x)Fxxx	Assumption
2	Faaa	A / ∃E
3	(∃z)Faaz	2 ∃I
4	(∃y)(∃z)Fayz	3 ∃I
5	(∃x)(∃y)(∃z)Fxyz	4 ∃I
6	(∃x)(∃y)(∃z)Fxyz	1, 2–5 ∃E

e. Derive: (∃x)(∃y)Gyx

1	(∀x)(Fx ⊃ (∃y)Gxy)	Assumption
2	(∃x)Fx	Assumption
3	Fa	A / ∃E
4	Fa ⊃ (∃y)Gay	1 ∀E
5	(∃y)Gay	3, 4 ⊃E
6	Gab	A /∃E
7	(∃y)Gyb	6 ∃I
8	(∃x)(∃y)Gyx	7 ∃I
9	(∃x)(∃y)Gyx	5, 6–8 ∃E
10	(∃x)(∃y)Gyx	2, 3–9 ∃E

g. Derive: (∃x)(∃y) ~ Hyx

1	(∀x)(∀y)(Hxy ⊃ ~ Hyx)	Assumption
2	(∃x)(∃y)Hxy	Assumption
3	(∃y)Hxa	A / ∃E
4	Hba	A / ∃E
5	(∀y)(Hby ⊃ ~ Hyb)	1 ∀E
6	Hba ⊃ ~ Hab	5 ∀E
7	~ Hab	4, 6 ⊃E
8	(∃y) ~ Hyb	7 ∃I
9	(∃x)(∃y) ~ Hyx	8 ∃I
10	(∃x)(∃y) ~ Hyx	3, 4–9 ∃E
11	(∃x)(∃y) ~ Hyx	2, 3–10 ∃E

i. Derive: (∀x)(∀y)Hxy

1	~ (∃x)(∃y)Rxy	Assumption
2	(∀x)(∀y)(~ Hxy ≡ Rxy)	Assumption
3	~ Hab	A / ~ E
4	(∀y)(~ Hay ≡ Ray)	2 ∀E
5	~ Hab ≡ Rab	4 ∀E
6	Rab	3, 5 ≡E
7	(∃y)Ray	6 ∃I
9	(∃x)(∃y)Rxy	7 ∃I
10	~ (∃x)(∃y)Rxy	1 R
11	Hab	3–10 ~ E
12	(∀y)Hay	11 ∀I
13	(∀x)(∀y)Hxy	12 ∀I

6. Validity

a. Derive: (∃y)Gya

1	(∀x)(Fx ⊃ Gba)	Assumption
2	(∃x)Fx	Assumption
3	Fb	A / ∃E
4	Fb ⊃ Gba	1 ∀E
5	Gba	3, 4 ⊃E
6	(∃y)Gya	5 ∃I
7	(∃y)Gya	2, 3–6 ∃E

c. Derive: (∃x) (∃y) Fxy

1	(∃x) (∃y) (Fxy ∨ Fyx)	Assumption
2	(∃y) (Fay ∨ Fya)	A / ∃E
3	Fab ∨ Fba	A / ∃E
4	Fab	A / ∨E
5	(∃y) Fay	4 ∃I
6	(∃x) (∃y) Fxy	5 ∃I
7	Fba	A /∨E
8	(∃y) Fby	7 ∃I
9	(∃x) (∃y) Fxy	8 ∃I
10	(∃x) (∃y) Fxy	3, 4–6, 7–9 ∨E
11	(∃x) (∃y) Fxy	2, 3–10 ∃E
12	(∃x) (∃y) Fxy	1, 2–11 ∃E

e. Derive: (∀z) (Faz ⊃ Fza)

1	(∀x) (∀y) [(∃z) [(Fyz & ~ Fzx) ⊃ Gxy]	Assumption
2	~ (∃x) Gxx	Assumption
3	Fab	A / ⊃I
4	~ Fba	A / ~ E
5	(∀y) [(∃z) (Fyz & ~ Fza) ⊃ Gay]	1 ∀E
6	(∃z) (Faz & ~ Fza) ⊃ Gaa	5 ∀E
7	Fab & ~ Fba	3, 4 &I
8	(∃z) (Faz & ~ Fza)	7 ∃I
9	Gaa	6, 8 ⊃E
10	(∃x) Gxx	9 ∃I
11	~ (∃x) Gxx	2 R
12	Fba	4–11 ~ E
13	Fab ⊃ Fba	3–12 ⊃I
14	(∀z) (Faz ⊃ Fza)	13 ∀I

g. Derive: (∀x) ~ Fx

1	(∀x)(Fx ⊃ (∃y)Gxy)	Assumption
2	(∀x)(∀y) ~ Gxy	Assumption
3	Fa	A / ~ I
4	Fa ⊃ (∃y)Gay	1 ∀E
5	(∃y)Gay	3, 4 ⊃E
6	Gab	A / ∃E
7	(∀x)(∀y) ~ Gxy	A / ~ I
8	(∀y) ~ Gay	2 ∀E
9	~ Gab	8 ∀E
10	Gab	6 R
11	~ (∀x)(∀y) ~ Gxy	7–11 ~ I
12	~ (∀x)(∀y) ~ Gxy	5, 6–11 ∃E
13	(∀x)(∀y) ~ Gxy	2 R
14	~ Fa	3–14 ~I
15	(∀x)~ Fx	14 ∀I

7. Theorems

a. Derive: (∀x)(∃z)(Fxz ⊃ Fzx)

1	Faa	A / ⊃I
2	Faa	1 R
3	Faa ⊃ Faa	1–2 ⊃I
4	(∃z)(Faz ⊃ Fza)	3 ∃I
5	(∀x)(∃z)(Fxz ⊃ Fzx)	4 ∀I

c. Derive: (∀x)(∀y)Gxy ⊃ (∀z)Gzz

1	(∀x)(∀y)Gxy	A / ⊃I
2	(∀y)Gay	1 ∀E
3	Gaa	2 ∀E
4	(∀z)Gzz	3 ∀I
5	(∀x)(∀y)Gxy ⊃ (∀z)Gzz	1–4 ⊃I

e. Derive: (∀x)Lxx ⊃ (∃x)(∃y)(Lxy & Lyx)

1	(∀x)Lxx	A / ⊃I
2	Laa	1 ∀E
3	Laa & Laa	2, 2 &I
4	(∃y)(Lay & Lya)	3 ∃I
5	(∃x)(∃y)(Lxy & Lyx)	4 ∃I
6	(∀x)Lxx ⊃ (∃x)(∃y)(Lxy & Lyx)	1–5 ⊃I

#h. The theorem to be proved, '(∃x) (∀y)Fxy ⊃ (∃x) (∃y)Fxy' is a truth-functional compound whose main connective is a material conditional. Therefore, we will use Conditional Introduction as our primary strategy:

```
Derive: (∃x) (∀y)Fxy ⊃ (∃x) (∃y)Fxy

1 |  | (∃x) (∀y)Fxy                         Assumption
  |  |-------------
  |  |
  |  |
G |  | (∃x) (∃y)Fxy
G | (∃x) (∀y)Fxy ⊃ (∃x) (∃y)Fxy             1–__ ⊃I
```

Our current goal is an existentially quantified sentence, '(∃x) (∃y)Fxy'. The most obvious way to obtain it is by two uses of Existential Introduction. Since the sentence on line 1 is an existentially quantified sentence it seems likely we will also be using Existential Elimination. And we know that when we do so, by assuming a substitution instance of '(∃x) (∀y)Fxy', we will have to continue working within that subderivation until we obtain a sentence that does not contain the instantiating constant. This suggests that our current goal, '(∃x) (∀y)Fxy', should also be the goal of our Existential Elimination subderivation, since it contains no constants:

```
Derive: (∃x) (∀y)Fxy ⊃ (∃x) (∃y)Fxy

1 |  | (∃x) (∀y)Fxy                         Assumption
  |  |-------------
2 |  |  | (∀y)Fay                           A / ∃E
  |  |  |--------
  |  |  |
  |  |  |
G |  |  | (∃x) (∃y)Fxy
G |  | (∃x) (∃y)Fxy                         2, 3–__ ∃E
G | (∃x) (∀y)Fxy ⊃ (∃x) (∃y)Fxy             1–__ ⊃I
```

Completing this derivation is now straightforward. We use Universal Elimination on line 2 to produce 'Fab' and then use Existential Introduction twice to produce '(∃x) (∃y)Fxy'.

Derive: $(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$

1	$(\exists x)(\forall y)Fxy$	Assumption
2	$(\forall y)Fay$	A / $\exists$E
3	Fab	2 $\forall$E
4	$(\exists y)Fay$	3 $\exists$I
5	$(\exists x)(\exists y)Fxy$	4 $\exists$I
6	$(\exists x)(\exists y)Fxy$	1, 2–5 $\exists$E
7	$(\exists x)(\forall y)Fxy \supset (\exists x)(\exists y)Fxy$	1–6 $\supset$I

Here we do meet all the restrictions on Existential Elimination. The instantiating constant, which is here ‘a’, does not, at the point we use Existential Elimination (line 6) occur in any open assumption. The constant ‘a’ also does not occur in the existentially quantified sentence to which we are applying Existential Elimination, and it does not occur in the sentence derived by Existential Elimination (the sentence on line 6).

It is worth noting that since there are no restrictions on Existential Introduction, we could have entered, at line 3, ‘Faa’ rather than ‘Fab’ (there are also no restrictions on Universal Elimination), and then twice applied Existential Introduction.

i. Derive: $(\exists x)(\exists y)(Lxy \equiv Lyx)$

1	Laa	A / $\equiv$I
2	Laa	1 R
3	$Laa \equiv Laa$	1–2, 1–2 $\equiv$I
4	$(\exists y)(Lay \equiv Lya)$	3 $\exists$I
5	$(\exists x)(\exists y)(Lxy \equiv Lyx)$	4 $\exists$I

k. Derive: $(\forall x)(\forall y)(\forall z)Gxyz \supset (\forall x)(\forall y)(\forall z)(Gxyz \supset Gzyx)$

1	$(\forall x)(\forall y)(\forall z)Gxyz$	A/ $\supset$I
2	$Gabc$	A / $\supset$I
3	$(\forall y)(\forall z)Gcyz$	1 $\forall$E
4	$(\forall z)Gcbz$	3 $\forall$E
5	$Gcba$	4 $\forall$E
6	$Gabc \supset Gcba$	2–5 $\supset$I
7	$(\forall z)(Gabz \supset Gzba)$	6 $\forall$I
8	$(\forall y)(\forall z)(Gayz \supset Gzya)$	7 $\forall$I
9	$(\forall x)(\forall y)(\forall z)(Gxyz \supset Gzyz)$	
10	$(\forall x)(\forall y)(\forall z)Gxyz \supset (\forall x)(\forall y)(\forall z)(Gxyz \supset Gzyx)$	1–9 $\supset$I

m. Derive: (∀x)(∀y)(Fxy ≡ Fyx) ⊃ ~ (∃x)(∃y)(Fxy & ~ Fyx)

1	(∀x)(∀y)(Fxy ≡ Fyx)	A / ⊃I
2	(∃x)(∃y)(Fxy & ~ Fyx)	A / ~ I
3	(∃y)(Fay & ~ Fya)	A / ∃E
4	Fab & ~ Fba	A / ∃E
5	(∀x)(∀y)(Fxy ≡ Fyx)	A / ~ I
6	(∀y)(Fay ≡ Fya)	1 ∀E
7	Fab ≡ Fba	6 ∀E
8	Fab	4 &E
9	Fba	7, 8 ≡E
10	~ Fba	4 &E
11	~ (∀x)(∀y)(Fxy ≡ Fyx)	5–10 ~ I
12	~ (∀x)(∀y)(Fxy ≡ Fyx)	3, 4–11 ∃E
13	~ (∀x)(∀y)(Fxy ≡ Fyx)	2, 3–12 ∃E
14	(∀x)(∀y)(Fxy ≡ Fyx)	1 R
15	~ (∃x)(∃y(Fxy & ~ Fyx)	2–14 ~ I
16	(∀x)(∀y)(Fxy ≡ Fyx) ⊃ ~ (∃x)(∃y)(Fxy & ~ Fyx)	1–15 ⊃I

8. Equivalence

a. Derive: (∀x)(Fx ⊃ (∃y)Gya)

1	(∃x)Fx ⊃ (∃y)Gya	Assumption
2	Fa	A / ⊃I
3	(∃x)Fx	2 ∃I
4	(∃y)Gya	1, 3 ⊃E
5	Fa ⊃ (∃y)Gya	2–4 ⊃I
6	(∀x)(Fx ⊃ (∃y)Gya)	5 ∀I

Derive: (∃x)Fx ⊃ (∃y)Gya

1	(∀x)(Fx ⊃ (∃y)Gya)	Assumption
2	(∃x)Fx	A / ⊃I
3	Fb	A / ∃E
4	Fb ⊃ (∃y)Gya	1 ∀E
5	(∃y)Gya	3, 4 ⊃E
6	(∃y)Gya	2, 3–5 ∃E
7	(∃x)Fx ⊃ (∃y)Gya	2–6 ⊃I

#c. To establish that '(∃x)[Fx ⊃ (∀y)Hxy]' and '(∃x)(∀y)(Fx ⊃ Hxy)' are equivalent in *PD* we have to derive each from the unit set of the other. We begin by deriving '(∃x)(∀y)(Fx ⊃ Hxy)' from {(∃x)[Fx ⊃ (∀y)Hxy]}. Since our one primary assumption will be an existentially quantified sentence we will use

Existential Elimination as our primary strategy and do virtually all of the derivation within that strategy:

Derive: (∃x)(∀y)(Fx ⊃ Hxy)

1	(∃x)[Fx ⊃ (∀y)Hxy]	Assumption
2	Fa ⊃ (∀y)Hay	A / ∃E
G	(∃x)(∀y)(Fx ⊃ Hxy)	I
G	(∃x)(∀y)(Fx ⊃ Hxy)	1, 2–__ ∃E

Our current goal is an existentially quantified sentence. We will try to obtain it by Existential Introduction, and will try to obtain the required substitution instance, which will be a universally quantified sentence, by Universal Introduction:

Derive: (∃x)(∀y)(Fx ⊃ Hxy)

1	(∃x)[Fx ⊃ (∀y)Hxy]	Assumption
2	Fa ⊃ (∀y)Hay	A / ∃E
3		
G	Fa ⊃ Hab	
G	(∀y)(Fa ⊃ Hay)	__ ∀I
G	(∃x)(∀y)(Fx ⊃ Hxy)	__ ∃I
G	(∃x)(∀y)(Fx ⊃ Hxy)	1, 2–__ ∃E

Our goal is now a material conditional, and we can obtain it by using Conditional Introduction and within that strategy Universal Elimination. The completed derivation is

```
Derive: (∃x) (∀y) (Fx ⊃ Hxy)

1  | (∃x) [Fx ⊃ (∀y)Hxy]              Assumption
   |---------------------
2  |   | Fa ⊃ (∀y)Hay                  A / ∃E
   |   |-----------------
3  |   |   | Fa                        A / ⊃I
   |   |   |-------------
4  |   |   | (∀y)Hay                   2, 3 ⊃E
5  |   |   | Hab                       4 ∀E
6  |   | Fa ⊃ Hab                      3–5 ⊃I
7  |   | (∀y) (Fa ⊃ Hay)               6 ∀I
8  |   | (∃x) (∀y) (Fx ⊃ Hxy)          7 ∃I
9  | (∃x) (∀y) (Fx ⊃ Hxy)              1, 2–8 ∃E
```

At line 5 we used Universal Elimination and in doing so were careful to pick an instantiating constant other than ‘a’ as our instantiating constant. Had we used ‘a’ we would not have been able to do Universal Introduction at line 7 because ‘a’ occurs in an assumption (the one on line 2) that is open as of line 7 and also occurs in line 7 itself.

When we apply Existential Elimination, at line 9, the instantiating constant, which is ‘a,’ does not occur in any open assumption, does not occur in the sentence we obtain at line 9, and of course does not occur in the existentially quantified sentence from which we are working (the sentence on line 1). So all three restrictions on Existential Elimination have been met. Note also that our use of Universal Introduction at line 7 meets both restrictions on that rule. The instantiating constant is ‘b’ and ‘b’ does not occur in any open assumption and does not occur in the sentence we obtain by Universal Introduction, ‘(∀y) (Fa ⊃ Hay)’

The derivation of ‘(∃x) [Fx ⊃ (∀y)Hxy]’ from {(∃x) (∀y) (Fx ⊃ Hxy)} is equally straightforward:

```
Derive: (∃x) [Fx ⊃ (∀y)Hxy]

1  | (∃x) (∀y) (Fx ⊃ Hxy)              Assumption
   |---------------------
2  |   | (∀y) (Fa ⊃ Hay)               A / ∃E
   |   |-----------------
3  |   |   | Fa                        A / ⊃I
   |   |   |-------------
4  |   |   | Fa ⊃ Hab                  2 ∀E
5  |   |   | Hab                       3, 4 ⊃E
6  |   |   | (∀y)Hay                   5 ∀I
7  |   | Fa ⊃ (∀y)Hay                  3–6 ⊃I
8  |   | (∃x) [Fx ⊃ (∀y)Hxy]           7 ∃I
9  | (∃x) [Fx ⊃ (∀y)Hxy]               1, 2–8 ∃E
```

We have again used Existential Elimination as our primary strategy and have again done the bulk of the work of the derivation within that strategy. We were again careful to pick an instantiating constant other than ‘a’ in doing Universal Elimination at line 4, again because using ‘a’ would prevent us from doing Universal Introduction at line 6.

e. Derive: (∀x) (∀y) (Fxy ≡ ~ Gyx)

1	(∀x) (∀y) ~ (Fxy ≡ Gyx)	Assumption
2	(∀y) ~ (Fay ≡ Gya)	1 ∀E
3	~ (Fab ≡ Gba)	2 ∀E
4	Fab	A / ≡I
5	Gba	A / ~ I
6	Fab	A / ≡I
7	Gab	5 R
8	Gab	A / ≡I
9	Fab	4 R
10	Fab ≡ Gab	6–7, 8–9 ≡I
11	~ (Fab ≡ Gab)	3 R
12	~ Gba	5–11 ~ I
13	~ Gba	A / ≡I
14	~ Fab	A / ~ E
15	Fab	A / ≡I
16	~ Gba	A / ~ I
17	Fba	15 R
18	~ Fba	14 R
19	Gba	16–18 ~ E
20	Gba	A / ≡I
21	~ Fba	A / ~ E
22	Gba	20 R
23	~ Gba	13 R
24	Fab	21–23 ~ E
25	Fab ≡ Gba	4–12, 13–24 ≡I
26	~ (Fab ≡ Gba)	3 R
27	Fab	14–26 ~ E
28	Fab ≡ ~ Gba	4–12, 13–27 ≡I
29	(∀y) (Fay ≡ ~ Gya)	28 ∀I
30	(∀x) (∀y) (Fxy ≡ ~ Gyx)	29 ∀I

Derive: (∀x)(∀y) ~ (Fxy ≡ Gyx)

1	(∀x)(∀y)(Fxy ≡ ~ Gyx)	Assumption
2	Fab ≡ Gba	A / ~ I
3	(∀y)(Fay ≡ ~ Gya)	1 ∀E
4	Fab ≡ ~ Gba	3 ∀E
5	Fab	A ≡I
6	~ Gba	4, 5 ≡E
7	Gba	2, 5 ≡E
8	~ Fab	5–7 ~ I
9	~ Gba	A / ~ E
10	Fab	4, 9 ≡E
11	Gba	2, 10 ≡E
12	~ Gba	9 R
13	Gba	9–12 ~ E
14	Fab	2, 13 ≡E
15	~ (Fab ≡ Gba)	2–14 ~ I
16	(∀y) ~ (Fay ≡ Gya)	15 ∀I
17	(∀x)(∀y) ~ (Fxy ≡ Gyx)	16 ∀I

9. Inconsistency

a. Derive: Tab, ~ Tab

b.

1	(∀x)(∀y)[(Ex & Ey) ⊃ Txy]	Assumption
2	(Ea & Eb) & ~ Tab	Assumption
3	(∀y)[(Ea & Ey) ⊃ Tay]	1 ∀E
4	(Ea & Eb) ⊃ Tab	3 ∀E
5	Ea & Eb	2 & E
6	Tab	4, 5 ⊃E
7	~ Tab	2 &E

c. Derive: (∃x)Fxx, ~ (∃x)Fxx

1	~ (∃x)Fxx	Assumption
2	(∃x)(∀y)Fxy	Assumption
3	(∀y)Fay	A / ∃E
4	Faa	3 ∀E
5	(∃x)Fxx	4 ∃I
6	(∃x)Fxx	2, 3–5 ∃E
7	~ (∃x)Fxx	1 R

e. Derive: (∀y) ~ Lay, ~ (∀y) ~ Lay

1	(∀x)(∃y)Lxy	Assumption
2	(∀y) ~ Lay	Assumption
3	(∃y)Lay	1 ∀E
4		Lab	A / ∃E
5			(∀y) ~ Lay	A / ~ I
6			~ Lab	6 ∀E
7			Lab	4 R
8		~ (∀y) ~ Lay	5–7 ~ I
9	~ (∀y) ~ Lay	3, 4–8 ∃E
10	(∀y) ~ Lay	2 R

g. Derive: (∃x) ~ (∃y)Lyx, ~ (∃x) ~ (∃y)Lyx

1	(∀x)[Hx ⊃ (∃y)Lyx]	Assumption
2	(∃x) ~ (∃y)Lyx	Assumption
3	(∀x)Hx	Assumption
4		~ (∃y)Lya	A / ∃E
5			(∃x) ~ (∃y)Lyx	A / ~ I
5			Ha ⊃ (∃y)Lya	1 ∀E
6			Ha	3 ∀E
7			(∃y)Lya	5, 6 ⊃E
8			~ (∃y)Lya	4 R
9		~ (∃x) ~ (∃y)Lyx	5–8 ~ I
10	~ (∃x) ~ (∃y)Lyx	2, 4–9 ∃E
11	(∃x) ~ (∃y)Lyx	2 R

#i. We will now show that the set {(∀x)(∃y)Fxy, (∃z) ~ (∃w)Fzw} is inconsistent in *PD*. This is an interesting problem in several respects. Neither set member is a negation. So it is not obvious which pair of contradictory sentences (the **Q** and ~ **Q** we must derive to show the set is contradictory) we should take as our goal. One of the set members is an existentially quantified sentence, so it is plausible that our derivation will involve an Existential Elimination as its main strategy, with a substitution instance of '(∃z) ~ (∃w)Fzw' as the assumption of a subderivation. Remembering that it is often useful to do as much of the work of a derivation as possible within an Existential Elimination subderivation we will make Existential Elimination our primary strategy:

Derive: ?, ?

1	(∀x)(∃y)Fxy	Assumption
2	(∃z) ~ (∃w)Fzw	Assumption
3		~ (∃w)Faw	A / ∃E

Our new assumption is a negation, but that is obviously no hope of moving that sentence out from within the scope of our subderivation so that it can play the role of ~ **Q** in our derivation – no hope because it obviously contains the instantiating constant ‘a’. A better strategy is to try to obtain a negation within the scope of the Existential Elimination strategy that does not contain the constant ‘a’. The obviously useful negation is ‘~ (∀x)(∃y)Fxy’ because we can obtain the sentence of which it is the negation, ‘(∀x)(∃y)Fxy’ by Reiteration on line 1. So we will proceed as follows:

Derive: (∀x)(∃y)Fxy, ~ (∀x)(∃y)Fxy

1	| (∀x)(∃y)Fxy	Assumption
2	| (∃z) ~ (∃w)Fzw	Assumption
3	| | ~ (∃w)Faw	A / ∃E
4	| | | (∀x)(∃y)Fxy	A / ~ I
G	| | ~ (∀x)(∃y)Fxy	__ –__ ~ I
G	| ~ (∀x)(∃y)Fxy	2, 3–__ ∃E
	| (∀x)(∃y)Fxy	1 R

We now need to derive a sentence and its negation within the scope of the assumption on line 4. There is no reason not to use the negation on line 3. We will do so, making our new goal ‘(∃w)Faw’:

Derive: (∀x)(∃y)Fxy, ~ (∀x)(∃y)Fxy

1	| (∀x)(∃y)Fxy	Assumption
2	| (∃z) ~ (∃w)Fzw	Assumption
3	| | ~ (∃w)Faw	A / ∃E
4	| | | (∀x)(∃y)Fxy	A / ~ I
G	| | | (∃w)Faw	
	| | | ~ (∃w)Faw	3 R
G	| | ~ (∀x)(∃y)Fxy	__ –__ ~ I
G	| ~ (∀x)(∃y)Fxy	2, 3–__ ∃E
	| (∀x)(∃y)Fxy	1 R

From line 1 we can obtain '(∃y)Fay' by Universal Elimination. And we can move from '(∃y)Fay' to '(∃w)Faw' by an Existential Elimination strategy. Our completed derivation is

Derive: (∀x)(∃y)Fxy, ~ (∀x)(∃y)Fxy

1	(∀x)(∃y)Fxy	Assumption
2	(∃z) ~ (∃w)Fzw	Assumption
3	~ (∃w)Faw	A / ∃E
4	(∀x)(∃y)Fxy	A / ~ I
5	(∃y)Fay	1 ∀E
6	Fab	A / ∃E
7	(∃w)Faw	6 ∃I
8	(∃w)Faw	5, 6–7 ∃E
9	~ (∃w)Faw	3 R
10	~ (∀x)(∃y)Fxy	4–9 ~ I
11	~ (∀x)(∃y)Fxy	2, 3–10 ∃E
12	(∀x)(∃y)Fxy	1 R

We have used Existential Elimination twice and in both instances we met all restrictions on that rule. In the first use, at line 8, the instantiating constant is 'b' and 'b' does not occur in either line 5 or line 8 and it does not, as of line 8, occur in any open assumption.

k. Derive: (∀x)(∀y)(Fxy ∨ Gxy), ~ (∀x)(∀y)(Fxy ∨ Gxy)

1	(∀x)(∀y)(Fxy ∨ Gxy)	Assumption
2	(∃x)(∃y)(~ Fxy & ~ Gxy)	Assumption
3	(∃y)(~ Fay & ~ Gay)	A / ∃E
4	~ Fab & ~ Gab	A / ∃E
5	(∀y)(Fay ∨ Gay)	1 ∀E
6	Fab ∨ Gab	5 ∀E
7	Fab	A / ∨E
8	(∀x)(∀y)(Fxy ∨ Gxy)	A / ~ I
9	Fab	7 R
10	~ Fab	4 &E
11	~ (∀x)(∀y)(Fxy ∨ Gxy)	8–10 ~ I
12	Gab	A ∨E
13	(∀x)(∀y)(Fxy ∨ Gxy)	A / ~ I
14	Gab	14 R
15	~ Gab	4 &E
16	~ (∀x)(∀y)(Fxy ∨ Gxy)	13–15 ~ I
17	~ (∀x)(∀y)(Fxy ∨ Gxy)	6, 7–11, 12–16 ∨E
18	~ (∀x)(∀y)(Fxy ∨ Gxy)	3, 4–17 ∃E
19	~ (∀x)(∀y)(Fxy ∨ Gxy)	2, 3–18 ∃E
20	(∀x)(∀y)(Fxy ∨ Gxy)	1 R

10.3E

1. Derivability

a. Derive: $(\exists y)(\sim Fy \vee \sim Gy)$

1	$\sim (\forall y)(Fy \;\&\; Gy)$	Assumption
2	$(\exists y) \sim (Fy \;\&\; Gy)$	1 QN
3	$(\exists y)(\sim Fy \vee \sim Gy)$	2 DeM

c. Derive: $(\exists z)(Az \;\&\; \sim Cz)$

1	$(\exists z)(Gz \;\&\; Az)$	Assumption
2	$(\forall y)(Cy \supset \sim Gy)$	Assumption
3	$\quad Gh \;\&\; Ah$	A / $\exists$E
4	$\quad Ch \supset \sim Gh$	2 $\forall$E
5	$\quad Gh$	3 &E
6	$\quad \sim \sim Gh$	5 DN
7	$\quad \sim Ch$	4, 6 MT
8	$\quad Ah$	3 &E
9	$\quad Ah \;\&\; \sim Ch$	8, 7 &I
10	$\quad (\exists z)(Az \;\&\; \sim Cz)$	9 $\exists$I
11	$(\exists z)(Az \;\&\; \sim Cz)$	1, 3–10 $\exists$E

e. Derive: $(\exists x)Cxb$

1	$(\forall x)[(\sim Cxb \vee Hx) \supset Lxx]$	Assumption
2	$(\exists y) \sim Lyy$	Assumption
3	$\quad \sim Lmm$	A / $\exists$E
4	$\quad (\sim Cmb \vee Hm) \supset Lmm$	1 $\forall$E
5	$\quad \sim (\sim Cmb \vee Hm)$	3, 4 MT
6	$\quad \sim \sim Cmb \;\&\; \sim Hm$	5 DeM
7	$\quad \sim \sim Cmb$	6 &E
8	$\quad Cmb$	7 DN
9	$\quad (\exists x)Cxb$	8 $\exists$I
10	$(\exists x)Cxb$	2, 3–9 $\exists$E

2. Validity

a. Derive: $(\forall y) \sim (Hby \vee Ryy)$

1	$(\forall y) \sim Jx$	Assumption
2	$(\exists y)(Hby \vee Ryy) \supset (\exists x)Jx$	Assumption
3	$\sim (\exists x)Jx$	1 QN
4	$\sim (\exists y)(Hby \vee Ryy)$	2, 3 MT
5	$(\forall y) \sim (Hby \vee Ryy)$	4 QN

c. Derive: (∀x)(∀y)Hxy & (∀x) ~ Tx

1	(∀x) ~ ((∀y)Hyx ∨ Tx)	Assumption
2	~ (∃y)(Ty ∨ (∃x) ~ Hxy)	Assumption
3	(∀y) ~ (Ty ∨ (∃x) ~ Hxy)	2 QN
4	~ (Ta ∨ (∃x) ~ Hxa)	3 ∀E
5	~ Ta & ~ (∃x) ~ Hxa	4 DeM
6	~ (∃x) ~ Hxa	5 &E
7	(∀x) ~ ~ Hxa	6 QN
8	~ ~ Hba	7 ∀E
9	Hba	8 DN
10	(∀y)Hby	9 ∀I
11	(∀x)(∀y)Hxy	10 ∀I
12	~ Ta	5 &E
13	(∀x) ~ Tx	12 ∀I
14	(∀x)(∀y)Hxy & (∀x) ~ Tx	11, 13 &I

e. Derive: (∃x) ~ Kxx

1	(∀z)[Kzz ⊃ (Mz & Nz)]	Assumption
2	(∃z) ~ Nz	Assumption
3	~ Ng	A / ∃E
4	Kgg ⊃ (Mg & Ng)	1 ∀E
5	~ Mg ∨ ~ Ng	3 ∨I
6	~ (Mg & Ng)	5 DeM
7	~ Kgg	4, 6 MT
8	(∃x) ~ Kxx	7 ∃I
9	(∃x) ~ Kxx	2, 3–8 ∃E

g. Derive: (∃w)(Gw & Bw) ⊃ (∀y)(Lyy ⊃ ~ Ay)

1	(∃z)Gz ⊃ (∀w)(Lww ⊃ ~ Hw)	Assumption
2	(∃x)Bx ⊃ (∀y)(Ay ⊃ Hy)	Assumption
3	(∃w)(Gw & Bw)	A / ⊃I
4	Gm & Bm	A / ∃E
5	Gm	4 &E
6	(∃z)Gz	5 ∃I
7	(∀w)(Lww ⊃ ~ Hw)	1, 6 ⊃E
8	Lcc ⊃ ~ Hc	7 ∀E
9	Bm	4 &E
10	(∃x)Bx	9 ∃I
11	(∀y)(Ay ⊃ Hy)	2, 10 ⊃E
12	Ac ⊃ Hc	11 ∀E
13	~ Hc ⊃ ~ Ac	12 Trans
14	Lcc ⊃ ~ Ac	8, 13 HS
15	(∀y)(Lyy ⊃ ~ Ay)	14 ∀I
16	(∀y)(Lyy ⊃ ~ Ay)	3, 4–15 ∃E
17	(∃w)(Gw & Bw) ⊃ (∀y)(Lyy ⊃ ~ Ay)	3–16 ⊃I

i. Derive: ~ (∀x)(∀y)Bxy ⊃ (∀x)(~ Gx ∨ ~ Hx)

1	~ (∀x)(~ Gx ∨ ~ Hx) ⊃ (∀x)[Cx & (∀y)(Ly ⊃ Axy)]	Assumption
2	(∃x) [Hx & (∀y)(Ly ⊃ Axy)] ⊃ (∀x)(Fx & (∀y)Bxy)	Assumption
3	~ (∀x)(~ Gx ∨ ~ Hx)	A / ⊃I
4	(∃x) ~ (~ Gx ∨ ~ Hx)	3 QN
5	~ (~ Gi ∨ ~ Hi)	A / ∃I
6	~ ~ Gi & ~ ~ Hi	5 DeM
7	~ ~ Hi	6 &E
8	Hi	7 DN
9	(∀x)[Cx & (∀y)(Ly ⊃ Axy)]	1, 3 ⊃E
10	Ci & (∀y)(Ly ⊃ Aiy)	9 ∀E
11	(∀y)(Ly ⊃ Aiy)	10 &E
12	Hi & (∀y)(Ly ⊃ Aiy)	8, 11 &I
13	(∃x)[Hx & (∀y)(Ly ⊃ Axy)]	12 ∃I
14	(∀x)(Fx & (∀y)Bxy)	2, 13 ⊃E
15	Fj & (∀y)Bjy	14 ∀E
16	(∀y)Bjy	15 &E
17	(∀x)(∀y)Bxy	16 ∀I
18	(∀x)(∀y)Bxy	4, 5–17 ∃E
19	~ (∀x)(~ Gx ∨ ~ Hx) ⊃ (∀x)(∀y)Bxy	3–18 ⊃I
20	~ (∀x)(∀y)Bxy ⊃ ~ ~ (∀x)(~ Gx ∨ ~ Hx)	19 Trans
21	~ (∀x)(∀y)Bxy ⊃ (∀x)(~ Gx ∨ ~ Hx)	20 DN

3. Theorems

a. Derive: (∀x)(Ax ⊃ Bx) ⊃ (∀x)(Bx ∨ ~ Ax)

1	(∀x)(Ax ⊃ Bx)	A / ⊃I
2	(∀x)(~ Ax ∨ Bx)	1 Impl
3	(∀x)(Bx ∨ ~ Ax)	2 Com
4	(∀x)(Ax ⊃ Bx) ⊃ (∀x)(Bx ∨ ~ Ax)	1–3 ⊃I

c. Derive: ~ (∃x)(Ax ∨ Bx) ⊃ (∀x) ~ Ax

1	~ (∃x)(Ax ∨ Bx)	A / ⊃I
2	(∀x) ~ (Ax ∨ Bx)	1 QN
3	~ (Ac ∨ Bc)	2 ∀E
4	~ Ac & ~ Bc	3 DeM
5	~ Ac	4 &E
6	(∀x) ~ Ax	5 ∀I
7	~ (∃x)(Ax ∨ Bx) ⊃ (∀x) ~ Ax	1–6 ⊃I

e. Derive: $((\exists x)Ax \supset (\exists x)Bx) \supset (\exists x)(Ax \supset Bx)$

1	$\sim (\exists x)(Ax \supset Bx)$	A / ⊃I
2	$(\forall x) \sim (Ax \supset Bx)$	1 QN
3	$\sim (Ac \supset Bc)$	2 ∀E
4	$\sim (\sim Ac \vee Bc)$	3 Impl
5	$\sim \sim Ac \;\&\; \sim Bc$	4 DeM
6	$\sim \sim Ac$	5 &E
7	$(\exists x) \sim \sim Ax$	6 ∃I
8	$\sim (\forall x) \sim Ax$	7 QN
9	$\sim \sim (\exists x)Ax$	8 QN
10	$\sim Bc$	5 &E
11	$(\forall x) \sim Bx$	10 ∀I
12	$\sim (\exists x)Bx$	11 QN
13	$\sim \sim (\exists x)Ax \;\&\; \sim (\exists x)Bx$	9, 12 &I
14	$\sim (\sim (\exists x)Ax \vee (\exists x)Bx)$	13 DeM
15	$\sim ((\exists x)Ax \supset (\exists x)Bx)$	14 Impl
16	$\sim (\exists x)(Ax \supset Bx) \supset \sim ((\exists x)Ax \supset (\exists x)Bx)$	1–15 ⊃I
17	$((\exists x)Ax \supset (\exists x)Bx) \supset (\exists x)(Ax \supset Bx)$	16 Trans

4. Equivalence

a. Derive: $(\exists x)(Ax \;\&\; \sim Bx)$

1	$\sim (\forall x)(Ax \supset Bx)$	Assumption
2	$(\exists x) \sim (Ax \supset Bx)$	1 QN
3	$(\exists x) \sim (\sim Ax \vee Bx)$	2 Impl
4	$(\exists x)(\sim \sim Ax \;\&\; \sim Bx)$	3 DeM
5	$(\exists x)(Ax \;\&\; \sim Bx)$	4 DN

Derive: $\sim (\forall x)(Ax \supset Bx)$

1	$(\exists x)(Ax \;\&\; \sim Bx)$	Assumption
2	$(\exists x)(\sim \sim Ax \;\&\; \sim Bx)$	1 DN
3	$(\exists x) \sim (\sim Ax \vee Bx)$	2 DeM
4	$(\exists x) \sim (Ax \supset Bx)$	3 Impl
5	$\sim (\forall x)(Ax \supset Bx)$	4 QN

c. Derive: $(\exists x)[\sim Ax \vee (\sim Cx \supset \sim Bx)]$

1	$\sim (\forall x) \sim [(Ax \;\&\; Bx) \supset Cx]$	Assumption
2	$(\exists x) \sim \sim [(Ax \;\&\; Bx) \supset Cx]$	1 QN
3	$(\exists x)[(Ax \;\&\; Bx) \supset Cx]$	2 DN
4	$(\exists x)[Ax \supset (Bx \supset Cx)]$	3 Exp
5	$(\exists x)[\sim Ax \vee (Bx \supset Cx)]$	4 Impl
6	$(\exists x)[\sim Ax \vee (\sim Cx \supset \sim Bx)]$	5 Trans

Derive: ~ (∀x) ~ [(Ax & Bx) ⊃ Cx]

1	(∃x)[~ Ax ∨ (~ Cx ⊃ ~ Bx)]	Assumption
2	(∃x)[~ Ax ∨ (Bx ⊃ Cx)]	1 Trans
3	(∃x)[Ax ⊃ (Bx ⊃ Cx)]	2 Impl
4	(∃x)[(Ax & Bx) ⊃ Cx]	3 Exp
5	~ ~ (∃x)[(Ax & Bx) ⊃ Cx]	4 DN
6	~ (∀x) ~ [(Ax & Bx) ⊃ Cx]	5 QN

e. Derive: ~ (∃x)[(~ Ax ∨ ~ Bx) & (Ax ∨ Bx)]

1	(∀x)(Ax ≡ Bx)	Assumption
2	~ ~ (∀x)(Ax ≡ Bx)	1 DN
3	~ (∃x) ~ (Ax ≡ Bx)	2 QN
4	~ (∃x) ~ [(Ax & Bx) ∨ (~ Ax & ~ Bx)]	3 Equiv
5	~ (∃x)[~ (Ax & Bx) & ~ (~ Ax & ~ Bx)]	4 DeM
6	~ (∃x)[(~ Ax ∨ ~ Bx) & ~ (~ Ax & ~ Bx)]	5 DeM
7	~ (∃x)[(~ Ax ∨ ~ Bx) & (~ ~ Ax ∨ ~ ~ Bx)]	6 DeM
8	~ (∃x)[(~ Ax ∨ ~ Bx) & (Ax ∨ ~ ~ Bx)]	7 DN
9	~ (∃x)[(~ Ax ∨ ~ Bx) & (Ax ∨ Bx)]	8 DN

Derive: (∀x)(Ax ≡ Bx)

1	~ (∃x)[(~ Ax ∨ ~ Bx) & (Ax ∨ Bx)]	Assumption
2	~ (∃x)[(~ Ax ∨ ~ Bx) & (Ax ∨ ~ ~ Bx)]	1 DN
3	~ (∃x)[(~ Ax ∨ ~ Bx) & (~ ~ Ax ∨ ~ ~ Bx)]	2 DN
4	~ (∃x)[(~ Ax ∨ ~ Bx) & ~ (~ Ax & ~ Bx)]	3 DeM
5	~ (∃x)[~ (Ax & Bx) & ~ (~ Ax & ~ Bx)]	4 DeM
6	~ (∃x) ~ [(Ax & Bx) ∨ (~ Ax & ~ Bx)]	5 DeM
7	~ (∃x) ~ (Ax ≡ Bx)	6 Equiv
8	~ ~ (∀x)(Ax ≡ Bx)	7 QN
9	(∀x)(Ax ≡ Bx)	8 DN

5. Inconsistency

a. Derive: Jc, ~ Jc

1	[(∀x)(Mx ≡ Jx) & ~ Mc] & (∀x)Jx	Assumption
2	(∀x)(Mx ≡ Jx) & ~ Mc	1 &E
3	(∀x)(Mx ≡ Jx)	2 &E
4	Mc ≡ Jc	3 ∀E
5	(Mc ⊃ Jc) & (Jc ⊃ Mc)	4 Equiv
6	Jc ⊃ Mc	5 &E
7	~ Mc	2 &E
8	~ Jc	6, 7 MT
9	(∀x)Jx	1 &E
10	Jc	9 ∀E

c. Derive: (∃w)Cww, ~ (∃w)Cww

1	(∀x)(∀y)Lxy ⊃ ~ (∃z)Tz	Assumption
2	(∀x)(∀y)Lxy ⊃ ((∃w)Cww ∨ (∃z)Tz)	Assumption
3	(~ (∀x)(∀y)Lxy ∨ (∀z)Bzzk) & (~ (∀z)Bzzk ∨ ~ (∃w)Cww)	Assumption
4	(∀x)(∀y)Lxy	Assumption
5	~ (∃z)Tz	1, 4 ⊃E
6	(∃w)Cww ∨ (∃z)Tz	2, 4 ⊃E
7	(∃w)Cww	5, 6 DS
8	~ (∀x)(∀y)Lxy ∨ (∀z)Bzzk	3 &E
9	(∀x)(∀y)Lxy ⊃ (∀z)Bzzk	8 Impl
10	(∀z)Bzzk	4, 9 ⊃E
11	~ (∀z)Bzzk ∨ ~ (∃w)Cww	3 &E
12	(∀z)Bzzk ⊃ ~ (∃w)Cww	11 Impl
13	~ (∃w)Cww	10, 12 ⊃E

e. Derive: Hc, ~ Hc

1	(∀x)(∀y)(Gxy ⊃ Hc)	Assumption
2	(∃x)Gix & (∀x)(∀y)(∀z)Lxyz	Assumption
3	~ Lcib ∨ ~ (Hc ∨ Hc)	Assumption
4	(∃x)Gix	2 &E
5	Gik	A / ⊃I
6	(∀y)(Giy ⊃ Hc)	1 ∀E
7	Gik ⊃ Hc	6 ∀E
8	Hc	5, 7 ⊃E
9	Hc	4, 5–8 ∃E
10	(∀x)(∀y)(∀z)Lxyz	2 &E
11	(∀y)(∀z)Lcyz	10 ∀E
12	(∀z)Lciz	11 ∀E
13	Lcib	12 ∀E
14	~ ~ Lcib	13 DN
15	~ (Hc ∨ Hc)	3, 14 DS
16	~ Hc	15 Idem

6. a. Suppose there is a sentence on an accessible line **i** of a derivation to which Universal Elimination can be properly applied at line **n**. The sentence that would be derived by Universal Elimination can also be derived by using the routine beginning at line **n**:

i	(∀**x**)**P**	
n	~ **P**(**a**/**x**)	A / ~ E
n + 1	(∃x) ~ **P**	**n** ∃I
n + 2	~ (∀**x**)**P**	**n** + 1 QN
n + 3	(∀**x**)**P**	**i** R
n + 4	**P**(**a**/**x**)	**n** − **n** + 3 ~ E

Suppose there is a sentence on an accessible line **i** of a derivation to which Universal Introduction can be properly applied at line **n**. The sentence that would be derived by Universal Introduction can also be derived by using the routine beginning at line **n**:

$\mathbf{i}$	$\mathbf{P(a/x)}$	
$\mathbf{n}$	$\sim (\forall \mathbf{x})\mathbf{P}$	A / ~ E
$\mathbf{n} + 1$	$(\exists \mathbf{x}) \sim \mathbf{P}$	**n** QN
$\mathbf{n} + 2$	$\sim \mathbf{P(a/x)}$	A / ~ E
$\mathbf{n} + 3$	$\sim (\forall \mathbf{x})\mathbf{P}$	A / ~ E
$\mathbf{n} + 4$	$\mathbf{P(a/x)}$	**i** R
$\mathbf{n} + 5$	$\sim \mathbf{P(a/x)}$	**n** + 2 R
$\mathbf{n} + 6$	$(\forall \mathbf{x})\mathbf{P}$	**n** + 3 − **n** + 5 ~ E
$\mathbf{n} + 7$	$(\forall \mathbf{x})\mathbf{P}$	**n** + 1, **n** + 2 − **n** + 6 ∃E
$\mathbf{n} + 8$	$\sim (\forall \mathbf{x})\mathbf{P}$	**n** R
$\mathbf{n} + 9$	$(\forall \mathbf{x})\mathbf{P}$	**n** − **n** + 8 ~ E

No restriction on the use of Existential Elimination was violated at line **n** + 7. We assumed that we could have applied Universal Introduction at line **n** to **P**(**a**/**x**) on line **i**. So **a** does not occur in any undischarged assumption prior to line **n**, and **a** does not occur in (∀**x**)**P**. So **a** does not occur in **P**. Hence

(i) **a** does not occur in any undischarged assumption prior to **n** + 7. Note that the assumptions on lines **n** + 2 and **n** + 3 have been discharged and that **a** cannot occur in the assumption on line **n**, for **a** does not occur in **P**.

(ii) **a** does not occur in (∃**x**) ~ **P**, for **a** does not occur in **P**.

(iii) **a** does not occur in (∀**x**)**P**, for **a** does not occur in **P**.

10.4E Exercises

1. Theorems

a. Derive: a = b ⊃ b = a

1	a = b	Assumption
2	a = a	1, 1 =E
3	b = a	1, 2 =E
4	a = b ⊃ b = a	1–3 ⊃I

c. Derive: (~ a = b & b = c) ⊃ ~ a = c

1	~ a = b & b = c	Assumption
2	~ a = b	1 &E
3	b = c	1 &E
4	~ a = c	2, 3 =E
5	(~ a = b & b = c) ⊃ ~ a = c)	1–4 ⊃I

e. Derive: ~ a = c ⊃ (~ a = b ∨ ~ b = c)

1	| | ~ a = c	Assumption
2	| | | ~ (~ a = b ∨ ~ b = c)	A / ~ E
3	| | | | ~ a = b	A / ~ E
4	| | | | ~ a = b ∨ ~ b = c	3 ∨I
5	| | | | ~ (~ a = b ∨ ~ b = c)	3 R
6	| | | a = b	3–5 ~ E
7	| | | ~ b = c	1, 6 =E
8	| | | ~ a = b ∨ ~ b = c	7 ∨I
9	| | | ~ (~ a = b ∨ ~ b = c)	2 R
10	| | ~ a = b ∨ ~ b = c	2–9 ~ E
11	| ~ a = c ⊃ (~ a = b ∨ ~ b = c)	1–10 ⊃I

2. Validity

a. Derive: ~ (∀x)Bxx

1	| a = b & ~ Bab	Assumption
2	| ~ Bab	1 &E
3	| a = b	1 &E
4	| | (∀x)Bxx	A / ~ I
5	| | Baa	4 ∀E
6	| | ~ Baa	2, 3 =E
7	| ~ (∀x)Bxx	4–6 ~ I

c. Derive: Hii

1	| (∀z)[Gz ⊃ (∀y)(Ky ⊃ Hzy)]	Assumption
2	| (Ki & Gj) & i = j	Assumption
3	| Gj ⊃ (∀y)(Ky ⊃ Hjy)	1 ∀E
4	| Ki & Gj	2 &E
5	| Gj	4 &E
6	| (∀y)(Ky ⊃ Hjy)	3, 5 ⊃E
7	| Ki ⊃ Hji	7 ∀E
8	| Ki	4 &E
9	| Hji	7, 8 ⊃E
10	| i = j	2 &E
11	| Hii	9, 10 =E

e. Derive: Ka ∨ ~ Kb

1	│ a = b	Assumption
2	│ │ ~ (Ka ∨ ~ Ka)	A / ~ E
3	│ │ │ Ka	A / ~ I
4	│ │ │ Ka ∨ ~ Ka	3 ∨I
5	│ │ │ ~ (Ka ∨ ~ Ka)	2 R
6	│ │ ~ Ka	3–5 ~ I
7	│ │ Ka ∨ ~ Ka	6 ∨I
8	│ │ ~ (Ka ∨ ~ Ka)	2 R
9	│ Ka ∨ ~ Ka	2–8 ~ E
10	│ Ka ∨ ~ Kb	1, 9 =E

3. Theorems

a. Derive: (∀x) (x = x ∨ ~ x = x)

1	│ (∀x)x = x	=I
2	│ a = a	1 ∀E
3	│ a = a ∨ ~ a = a	2 ∨I
4	│ (∀x) (x = x ∨ ~ x = x)	3 ∀I

c. Derive: (∀x) (∀y) (x = y ≡ y = x)

1	│ │ a = b	A / ≡I
2	│ │ a = a	1, 1 =E
3	│ │ b = a	1, 2 =E
4	│ │ b = a	A / ≡I
5	│ │ b = b	4, 4 =E
6	│ │ a = b	4, 5 =E
7	│ a = b ≡ b = a	1–3, 4–6 ≡I
8	│ (∀y) (a = y ≡ y = a)	7 ∀I
9	│ (∀x) (∀y) (x = y ≡ y = x)	8 ∀I

e. Derive: ~ (∃x) ~ x = x

1	│ │ (∃x) ~ x = x	A / ~ I
2	│ │ │ ~ a = a	A / ∃E
3	│ │ │ │ (∃x) ~ x = x	A / ~ I
4	│ │ │ │ (∀x)x = x	=I
5	│ │ │ │ a = a	4 ∀E
6	│ │ │ │ ~ a = a	2 R
7	│ │ │ ~ (∃x) ~ x = x	3–6, ~ I
8	│ │ ~ (∃x) ~ x = x	1, 2–7 ∃E
9	│ │ (∃x) ~ x = x	1 R
10	│ ~ (∃x) ~ x = x	1–9 ~ I

4. Validity

a. Derive: (∃x) (∃y) [(Ex & Ey) & ~ x = y]

1	~ t = f	Assumption
2	Et & Ef	Assumption
3	(Et & Ef) & ~ t = f	1, 2 &I
4	(∃y) [(Et & Ey) & ~ t = y]	3 ∃I
5	(∃x) (∃y) [(Ex & Ey) & ~ x = y]	4 ∃I

c. Derive: ~ s = b

1	~ Ass & Aqb	Assumption
2	(∀x) [(∃y)Ayx ⊃ Abx]	Assumption
3	s = b	A / ~ I
4	(∃y)Ayb ⊃ Abb	2 ∀E
5	Aqb	1 &E
6	(∃y)Ayb	5 ∃I
7	Abb	4, 6 ⊃E
8	~ Ass	1 &E
9	~ Abb	3, 8 =E
10	~ s = b	3–9 ~ I

e. Derive: (∃x) [(Rxe & Pxa) & (~ x = e & ~ x = a)]

1	(∃x) (Rxe & Pxa)	Assumption
2	~ Ree	Assumption
3	~ Paa	Assumption
4	Rie & Pia	A / ∃E
5	i = e	A / ~ I
6	Rie	4 &E
7	Ree	5, 6 =E
8	~ Ree	2 R
9	~ i = e	5–8 ~ I
10	i = a	A / ~ I
11	Pia	4 &E
12	Paa	10, 11 =E
13	~ Paa	3 R
14	~ i = a	10–13 ~ I
15	~ i = e & ~ i = a	9, 14 &I
16	(Rie & Pia) & (~ i = e & ~ i = a)	4, 15 &I
17	(∃x) [(Rxe & Pxa) & (~ x = e & ~ x = a)]	16 ∃I
18	(∃x) [(Rxe & Pxa) & (~ x = e & ~ x = a)]	1, 4–17 ∃E

5. a.

1	$(\exists x)Sx$	Assumption
2	$\quad Sg(f)$	A / $\exists$E
3	$\quad (\exists x)Sg(x)$	2 $\exists$I
4	$(\exists x)Sg(x)$	1, 2–3 $\exists$E

Line 2 is a mistake as an instantiating individual constant must be used, *not* a closed complex term.

c. Correctly done.

e.

1	$(\forall x)Lxxx$	Assumption
2	$Lf(a,a)a$	1 $\forall$E
3	$(\forall x)Lf(x,x)x$	2 $\forall$I

Line 2 is a mistake. Universal Elimination does not permit using both a closed complex term and at the same time an individual constant in the substitution instance, not to mention that all three occurrences of the variable ‘x’ must be replaced.

g.

1	$(\forall x)Rf(x,x)$	Assumption
2	$Rf(c,c)$	1 $\forall$E
3	$(\forall y)Ry$	2 $\forall$I

Line 3 is a mistake. Universal Introduction cannot be applied using a closed complex term.

i. Correctly done.

6. Theorems in *PDE*:

a. Derive: $(\forall x)(\exists y)f(x) = y$

1	$(\forall x)x = x$	=I
2	$f(a) = f(a)$	1 $\forall$E
3	$(\exists y)f(a) = y$	2 $\exists$I
4	$(\forall x)(\exists y)f(x) = y$	3 $\forall$I

c. Derive: $(\forall x)Ff(x) \supset (\forall x)Ff(g(x))$

1	$\quad (\forall x)Ff(x)$	A / $\supset$I
2	$\quad Ff(g(a))$	1 $\forall$E
3	$\quad (\forall x)Ff(g(x))$	2 $\forall$I
4	$(\forall x)Ff(x) \supset (\forall x)Ff(g(x))$	1–3 $\supset$I

e. Derive: (∀x)(f(f(x)) = x ⊃ f(f(f(f(x)))) = x)

1	| f(f(a)) = a	A / ⊃I
2	| f(f(f(f(a)))) = a	1, 1 =E
3	f(f(a)) = a ⊃ f(f(f(f(a)))) = a	1–2 ⊃I
4	(∀x)(f(f(x)) = x ⊃ f(f(f(f(x)))) = x)	3 ∀I

g. Derive: (∀x)(∀y)[(f(x) = y & f(y) = x) ⊃ x = f(f(x))]

1	| f(a) = b & f(b) = a	A / ⊃ I
2	| f(b) = a	1 &E
3	| f(b) = f(b)	2, 2 =E
4	| a = f(b)	2, 3 =E
5	| f(a) = b	1 &E
6	| a = f(f(a))	4, 5 =E
7	(f(a) = b & f(b) = a) ⊃ a = f(f(a))	1–6 ⊃ I
8	(∀y)[(f(a) = y & f(y) = a) ⊃ a = f(f(a))]	7 ∀I
9	(∀x)(∀y)[(f(x) = y & f(y) = x) ⊃ x = f(f(x))]	8 ∀I

7. Validity in *PDE*:

a. Derive: (∀x)Gf(x)f(f(x))

1	(∀x)(Bx ⊃ Gxf(x))	Assumption
2	(∀x)Bf(x)	Assumption
3	Bf(a) ⊃ Gf(a)f(f(a))	1 ∀E
4	Bf(a)	2 ∀I
5	Gf(a)f(f(a))	3, 4 ⊃E
6	(∀x)Gf(x)f(f(x))	5 ∀I

c. Derive: ~ f(a) = b

1	(∀x)(∀y)(f(x) = y ⊃ Myxc)	Assumption
2	~ Mbac & ~ Mabc	Assumption
3	(∀y)(f(a) = y ⊃ Myac)	1 ∀E
4	f(a) = b ⊃ Mbac	3 ∀E
5	| f(a) = b	A / ~ I
6	| Mbac	4, 5 ⊃E
7	| ~ Mbac	2 &E
8	~ f(a) = b	5–7 ~ I

e. Derive: $(\exists x)Lxf(x)g(x)$

1	$(\exists x)(\forall y)(\forall z)Lxyz$	Assumption
2	$(\forall y)(\forall z)Layz$	A / $\exists$E
3	$(\forall z)Laf(a)z$	2 $\forall$E
4	$Laf(a)g(a)$	3 $\forall$E
5	$(\exists x)Lxf(x)g(x)$	4 $\exists$I
6	$(\exists x)Lxf(x)g(x)$	1, 2–5 $\exists$E

g. Derive: $(\forall x)Df(x)f(x)$

1	$(\forall x)[Zx \supset (\forall y)(\sim Dxy \equiv Hf(f(y)))]$	Assumption
2	$(\forall x)(Zx \;\&\; \sim Hx)$	Assumption
3	$Zf(a) \supset (\forall y)(\sim Df(a)y \equiv Hf(f(y)))$	1 $\forall$E
4	$Zf(a) \;\&\; \sim Hf(a)$	2 $\forall$E
5	$Zf(a)$	4 &E
6	$(\forall y)(\sim Df(a)y \equiv Hf(f(y)))$	3, 5 $\supset$E
7	$\sim Df(a)f(a) \equiv Hf(f(f(a)))$	6 $\forall$E
8	$\sim Df(a)f(a)$	A / $\sim$ E
9	$Hf(f(f(a)))$	7, 8 $\equiv$E
10	$Zf(f(f(a))) \;\&\; \sim Hf(f(f(a)))$	2 $\forall$E
11	$\sim Hf(f(f(a)))$	10 &E
12	$Df(a)f(a)$	8–11 $\sim$ E
13	$(\forall x)Df(x)f(x)$	12 $\forall$I