Section 6.1E

1.a. We shall prove that every sentence of *SL* that contains only binary connectives, if any, is true on every truth-value assignment on which all its atomic components are true. Hence every sentence of *SL* that contains only binary connectives is true on at least one truth-value assignment, and thus no such sentence can be truth-functionally false. We proceed by mathematical induction on the number of occurrences of connectives in such sentences. (Note that we need not consider *all* sentences of *SL* in our induction but only those with which the thesis is concerned.)

Basis clause: Every sentence with zero occurrences of a binary connective (and no occurrences of unary connectives) is true on every truth-value assignment on which all its atomic components are true.

Inductive step: If every sentence with **k** or fewer occurrences of binary connectives (and no occurrences of unary connectives) is true on every truth-value assignment on which all its atomic components are true, then every sentence with **k** + 1 occurrences of binary connectives (and no occurrences of unary connectives) is true on every truth-value assignment on which all its atomic components are true.

The proof of the basis clause is straightforward. A sentence with zero occurrences of a connective is an atomic sentence, and each atomic sentence is true on every truth-value assignment on which its atomic component (which is the sentence itself) is true.

The inductive step is also straightforward. Assume that the thesis holds for every sentence of *SL* with **k** or fewer occurrences of binary connectives and no unary connectives. Any sentence **P** with **k** + 1 occurrences of binary connectives and no unary connectives must be of one of the four forms **Q** & **R**, **Q** ∨ **R**, **Q** ⊃ **R**, and **Q** ≡ **R**. In each case **Q** and **R** contain **k** or fewer occurrences of binary connectives, so the inductive hypothesis holds for both **Q** and **R**. That is, both **Q** and **R** are true on every truth-value assignment on which all their atomic components are true. Since **P**'s immediate components are **Q** and **R**, its atomic components are just those of **Q** and **R**. But conjunctions, disjunctions, conditionals, and biconditionals are true when both their immediate components are true. So **P** is also true on every truth-value assignment on which its atomic components are true, for both its immediate components are then true. This completes our proof. (Note that in this clause we ignored sentences of the form ~ **Q**, for the thesis concerns only those sentences of *SL* that contain *no* occurrences of '~'.)

b. Every sentence **P** that contains no binary connectives either contains no connectives or contains at least one occurrence of '~'. We prove the thesis by mathematical induction on the number of occurrences of '~' in such

sentences. The first case consists of the atomic sentences of *SL* since these contain zero occurrences of connectives.
Basis clause: Every atomic sentence is truth-functionally indeterminate.
Inductive step: If every sentence with **k** or fewer occurrences of '~' (and no binary connectives) is truth-functionally indeterminate, then every sentence with **k** + 1 occurrences of '~' (and no binary connectives) is truth-functionally indeterminate.

The basis clause is obvious.

The inductive step is also obvious. Suppose **P** contains **k** + 1 occurrences of '~' and no binary connectives and that the thesis holds for every sentence with fewer than **k** + 1 occurrences of '~' and no binary connectives. **P** is a sentence of the form ~ **Q**, where **Q** contains **k** occurrences of '~'; hence, by the inductive hypothesis, **Q** is truth-functionally indeterminate. The negation of a truth-functionally indeterminate sentence is also truth-functionally indeterminate. Hence ~ **Q**, that is, **P**, is truth-functionally indeterminate. This completes the induction.

c. The induction is on the number of occurrences of connectives in **P**. The thesis to be proved is

> If two truth-value assignments **A′** and **A″** assign the same truth-values to the atomic components of a sentence **P**, then **P** has the same truth-value on **A′** and **A″**.

Basis clause: The thesis holds for every sentence with zero occurrences of connectives.
Inductive step: If the thesis holds for every sentence with **k** or fewer occurrences of connectives, then the thesis holds for every sentence with **k** + 1 occurrences of connectives.

The basis clause is obvious. If **P** contains zero occurrences of connectives, then **P** is an atomic sentence and its own only atomic component. **P** must have the same truth-value on **A′** and **A″** because *ex hypothesi* it is assigned the same truth-value on each assignment.

To prove the inductive step, we let **P** be a sentence with **k** + 1 occurrences of connectives and assume that the thesis holds for every sentence containing **k** or fewer occurrences of connectives. Then **P** is of the form ~ **Q**, **Q** & **R**, **Q** ∨ **R**, **Q** ⊃ **R**, or **Q** ≡ **R**. In each case the immediate component(s) of **P** contain **k** or fewer occurrences of connectives and hence fall under the inductive hypothesis. So each immediate component of **P** has the same truth-value on **A′** and **A″**. **P** therefore has the same truth-value on **A′** and **A″**, as determined by the characteristic truth-tables.

d. We prove the thesis by mathematical induction on the number of conjuncts in an iterated conjunction of sentences $\mathbf{P}_1, \ldots, \mathbf{P}_\mathbf{n}$ of *SL*.
Basis clause: Every iterated conjunction of just one sentence of *SL* is true on a truth-value assignment if and only if that one sentence is true on that assignment.
Inductive step: If every iterated conjunction of **k** or fewer sentences of *SL* is true

on a truth-value assignment if and only if each of those conjuncts is true on that assignment, then every iterated conjunction of **k** + 1 sentences of *SL* is true on a truth-value assignment if and only if each of those conjuncts is true on that assignment.

The basis clause is trivial.

To prove the inductive step, we assume that the thesis holds for iterated conjunctions of **k** or fewer sentences of *SL*. Let **P** be an iterated conjunction of **k** + 1 sentences. Then **P** is **Q** & **R**, where **Q** is an iterated conjunction of **k** sentences. **P** is therefore an iterated conjunction of all the sentences of which **Q** is an iterated conjunction, and **R**. By the inductive hypothesis, the thesis holds of **Q**; that is, **Q** is true on a truth-value assignment if and only if the sentences of which **Q** is an iterated conjunction are true on that assignment. Hence, whenever all the sentences of which **P** is an iterated conjunction are true, both **Q** and **R** are true, and thus **P** is true as well. Whenever at least one of those sentences is false, either **Q** is false or **R** is false, making **P** false as well. Hence **P** is true on a truth-value assignment if and only if all the sentences of which it is an iterated conjunction are true on that assignment.

e. We proceed by mathematical induction on the number of occurrences of connectives in **P**. The argument is

> The thesis holds for every atomic sentence **P**.
>
> If the thesis holds for every sentence **P** with **k** or fewer occurrences of connectives, then it holds for every sentence **P** with **k** + 1 occurrences of connectives.
>
> ---
>
> The thesis holds for every sentence **P** of *SL*.

The proof of the basis clause is fairly simple. If **P** is an atomic sentence and **Q** is a sentential component of **P**, then **Q** must be identical with **P** (since each atomic sentence is its own only atomic component). For any sentence $\mathbf{Q}_1$, then, [**P**]($\mathbf{Q}_1$//**Q**) is simply the sentence $\mathbf{Q}_1$. Here it is trivial that if **Q** and $\mathbf{Q}_1$ are truth-functionally equivalent, so are **P** (which is just **Q**) and [**P**]($\mathbf{Q}_1$//**Q**) (which is just $\mathbf{Q}_1$).

In proving the inductive step, the following result will be useful:

6.1.1. If **Q** and $\mathbf{Q}_1$ are truth-functionally equivalent and **R** and $\mathbf{R}_1$ are truth-functionally equivalent, then each of the following pairs are pairs of truth-functionally equivalent sentences:

$\sim \mathbf{Q}$	$\sim \mathbf{Q}_1$
$\mathbf{Q} \,\&\, \mathbf{R}$	$\mathbf{Q}_1 \,\&\, \mathbf{R}_1$
$\mathbf{Q} \vee \mathbf{R}$	$\mathbf{Q}_1 \vee \mathbf{R}_1$
$\mathbf{Q} \supset \mathbf{R}$	$\mathbf{Q}_1 \supset \mathbf{R}_1$
$\mathbf{Q} \equiv \mathbf{R}$	$\mathbf{Q}_1 \equiv \mathbf{R}_1$

Proof: The truth-value of a molecular sentence is wholly determined by the truth-values of its immediate components. Hence, if there is a truth-value assignment on which some sentence in the left-hand column has a truth-value different from that of its partner in the right-hand column, then on that assignment either **Q** and $\mathbf{Q}_1$ have different truth-values or **R** and $\mathbf{R}_1$ have different truth-values. But this is impossible because *ex hypothesi* **Q** and $\mathbf{Q}_1$ are truth-functionally equivalent and **R** and $\mathbf{R}_1$ are truth-functionally equivalent.

To prove the inductive step of the thesis, we assume the inductive hypothesis: that the thesis holds for every sentence with **k** or fewer occurrences of connectives. Let **P** be a sentence of *SL* with **k** + 1 occurrences of connectives, let **Q** be a sentential component of **P**, let $\mathbf{Q}_1$ be a sentence that is truth-functionally equivalent to **Q**, and let $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ be a sentence that results from replacing one or more occurrences of **Q** in **P** with $\mathbf{Q}_1$. Suppose, first, that **Q** is identical with **P**. Then, by the reasoning in the proof of the basis clause, it follows trivially that **P** and $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ are truth-functionally equivalent. Now suppose that **Q** is a sentential component of **P** that is *not* identical with **P** (in which case we say that **Q** is a *proper* sentential component of **P**). Either **P** is of the form ~ **R** or **P** has a binary connective as its main connective and is of one of the four forms **R** & **S**, **R** ∨ **S**, **R** ⊃ **S**, and **R** ≡ **S**. We shall consider the two cases separately.

i. **P** is of the form ~ **R**. Since **Q** is a proper sentential component of **P**, **Q** must be a sentential component of **R**. Hence $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ is a sentence ~ $[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$. But **R** has **k** occurrences of connectives, so by the inductive hypothesis, **R** is truth-functionally equivalent to $[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$. It follows from 6.1.1 that ~ **R** is truth-functionally equivalent to ~ $[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$; that is, **P** is truth-functionally equivalent to $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$.

ii. **P** is of the form **R** & **S**, **R** ∨ **S**, **R** ⊃ **S**, or **R** ≡ **S**. Since **Q** is a proper component of **P**, $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ must be **P** with its left immediate component replaced by a sentence $[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$, **P** with its right immediate component replaced with a sentence $[\mathbf{S}](\mathbf{Q}_1//\mathbf{Q})$, or **P** with both replacements made. Both **R** and **S** have fewer than **k** + 1 occurrences of connectives, and so the inductive hypothesis holds for both **R** and **S**. Hence **R** is truth-functionally equivalent to $[\mathbf{R}](\mathbf{Q}_1//\mathbf{Q})$, and **S** is truth-functionally equivalent to $[\mathbf{S}](\mathbf{Q}_1//\mathbf{Q})$. And **R** is truth-functionally equivalent to **R** and **S** is truth-functionally equivalent to **S**. Whatever replacements are made in **P**, it follows by 6.1.1 that **P** is truth-functionally equivalent to $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$.

This completes the proof of the inductive step and thus the proof of our thesis.

2. An example of a sentence that contains only binary connectives and is truth-functionally true is 'A ⊃ A'. An attempted proof would break down in the proof of the inductive step (since no atomic sentence is truth-functionally true, the basis clause will go through).

Section 6.2E

1. Suppose that we have constructed, in accordance with the algorithm, a sentence for a row of a truth-function schema that defines a truth-function of **n** arguments. We proved in Exercise 1.d in Section 6.1E the result that an iterated conjunction (. . . ($\mathbf{P}_1$ & $\mathbf{P}_2$) & . . . & $\mathbf{P}_\mathbf{n}$) is true on a truth-value assignment if and only if $\mathbf{P}_1$, . . . , $\mathbf{P}_\mathbf{n}$ are all true on that truth-value assignment. We have constructed the present iterated conjunction of atomic sentences and negations of atomic sentences in such a way that each conjunct is true when the atomic components have the truth-values represented in that row. Hence for that assignment the sentence constructed is true. For any other assignments to the atomic components of the sentence, at least one of the conjuncts is false; hence the conjunction is also false.

2.a. (A & ~ B) ∨ (~ A & ~ B)
b. A & ~ A
d. ([(A & B) & C] ∨ [(A & B) & ~ C]) ∨ [(~ A & ~ B) & C]

3. Suppose that the table defines a truth-function of **n** arguments. We first construct an iterated disjunction of **n** disjuncts such that the **i**th disjunct is the negation of the **i**th atomic sentence of *SL* if the **i**th truth-value in the row is **T**, and the **i**th disjunct is the **i**th atomic sentence of *SL* if the **i**th truth-value in the row is **F**. Note that this iterated disjunction is *false* exactly when its atomic components have the truth-values displayed in that row. We then negate the iterated disjunction, to obtain a sentence that is *true* for those truth-values and false for all other truth-values that may be assigned to its atomic components.

4. To prove that {'~', '&'} is truth-functionally complete, it will suffice to show that for each sentence of *SL* containing only '~', '∨', and '&', there is a truth-functionally equivalent sentence of *SL* that contains the same atomic components and in which the only connectives are '~' and '&'. For it will then follow, from the fact that {'~', '∨', '&'} is truth-functionally complete, that {'~', '&'} is also truth-functionally complete. But every sentence of the form

P ∨ **Q**

is truth-functionally equivalent to

~ (~ **P** & ~ **Q**)

So by repeated substitutions, we can obtain, from sentences containing '~', '∨', and '&', truth-functionally equivalent sentences that contain only '~' and '&'.

To show that {'~', '⊃'} is truth-functionally complete, it suffices to point out that every sentence of the form

P & **Q**

is truth-functionally equivalent to the corresponding sentence

$$\sim (\mathbf{P} \supset \sim \mathbf{Q})$$

and that every sentence of the form

$$\mathbf{P} \vee \mathbf{Q}$$

is truth-functionally equivalent to the corresponding sentence

$$\sim \mathbf{P} \supset \mathbf{Q}$$

For then we can find, for each sentence containing only '~', '∨', and '&', a truth-functionally equivalent sentence with the same atomic components containing only '~' and '⊃'. It follows that {'~', '⊃'} is truth-functionally complete, since {'~', '∨', '&'} is.

5. To show this, we need only note that the negation and disjunction truth-functions can be expressed using only the dagger. The truth-table for 'A ↓ A' is

A	A	↓	A
T	**T**	**F**	**T**
F	**F**	**T**	**F**

The sentence 'A ↓ A' expresses the negation truth-function, for the column under the dagger is identical with the column to the right of the vertical line in the characteristic truth-table for negation.

The disjunction truth-function is expressed by '(A ↓ B) ↓ (A ↓ B)', as the following truth-table shows:

A	B	(A	↓	B)	↓	(A	↓	B)
T	**T**	**T**	**F**	**T**	**T**	**T**	**F**	**T**
T	**F**	**T**	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**F**	**F**	**T**	**T**	**F**	**F**	**T**
F	**F**	**F**	**T**	**F**	**F**	**F**	**T**	**F**

This table shows that '(A ↓ B) ↓ (A ↓ B)' is true on every truth-value assignment on which at least one of 'A' and 'B' is true. Hence that sentence expresses the disjunction truth-function.

Thus any truth-function that is expressed by a sentence of *SL* containing only the connectives '~' and '∨' can be expressed by a sentence containing only '↓' as a connective. To form such a sentence, we convert the sentence of *SL* containing just '~' and '∨' that expresses the truth-function in question as follows. Repeatedly replace components of the form ~ **P** with **P** ↓ **P** and components of the form **P** ∨ **Q** with (**P** ↓ **Q**) ↓ (**P** ↓ **Q**) until a sentence containing '↓' as the only connective is obtained. Since {'∨', '~'} is truth-functionally complete, so is {'↓'}.

7. The set {'~'} is not truth-functionally complete because every sentence containing only '~' is truth-functionally indeterminate. Hence truth-functions expressed in *SL* by truth-functionally true sentences and truth-functions expressed in *SL* truth-functionally false sentences cannot be expressed by a sentence that contains only '~'.

The set {'&', '∨', '⊃', '≡'} is not truth-functionally complete because no sentence that contains only binary connectives (if any) is truth-functionally false. Hence no truth-function that is expressed in *SL* by a truth-functionally false sentence can be expressed by a sentence containing only binary connectives of *SL*.

8. We shall prove by mathematical induction that in the truth-table for a sentence **P** containing only the connectives '~' and '≡' and two atomic components, the column under the main connective of **P** has an even number of **T**s and an even number of **F**s. For then we shall know that no sentence containing only those connectives can express, for example, the truth-function defined as follows (the material conditional truth-function):

T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

In the induction remember that any sentence of *SL* that contains two atomic components has a four-row truth-table. Our induction will proceed on the number of occurrences of connectives in **P**. However, the first case, that considered in the basis clause, is the case where **P** contains *one* occurrence of a connective. This is because every sentence that contains zero occurrences of connectives is an atomic sentence and thus cannot contain more than one atomic component.

Basis clause: The thesis holds for every sentence of *SL* with exactly two atomic components and one occurrence of (one of) the connectives '~' and '≡'.

In this case **P** cannot be of the form ~ **Q**, for if the initial '~' is the only connective in **P**, then **Q** is atomic, and hence **P** does not contain two atomic components. So **P** is of the form **Q** ≡ **R**, where **Q** and **R** are atomic sentences. **Q** ≡ **R** will have to be true on assignments that assign the same truth-values to **Q** and **R** and false on other assignments. Hence the thesis holds in this case.

Inductive step: If the thesis holds for every sentence of *SL* that contains **k** or fewer occurrences of the connectives '~' and '≡' (and no other connectives) and two atomic components, then the thesis holds for every sentence of *SL* that contains two atomic components and **k** + 1 occurrences of the connectives '~' and '≡' (and no other connectives).

Let **P** be a sentence of *SL* that contains exactly two atomic components and **k** + 1 occurrences of the connectives '~' and '≡' (and no other connectives). There are two cases to consider.

i. **P** is of the form ~ **Q**. Then **Q** falls under the inductive hypothesis; hence in the truth-table for **Q** the column under the main connective contains an even number of **T**s and an even number of **F**s. The column for the sentence ~ **Q** simply reverses the **T**s and **F**s, so it also contains an even number of **T**s and an even number of **F**s.

ii. **P** is of the form **Q** ≡ **R**. Then **Q** and **R** each contain fewer occurrences of connectives. If, in addition, **Q** and **R** each contain both of the atomic components of **P**, then they fall under the inductive hypothesis—**Q** has an even number of **T**s and an even number of **F**s in its truth-table column, and so does **R**. On the other hand, if **Q** or **R** (or both) only contains one of the atomic components of **P** (e.g., if **P** is '~ A ≡ (B ≡ A)' then **Q** is '~ A'), then **Q** or **R** (or both) fails to fall under the inductive hypothesis. However, in this case the component in question also has an even number of **T**s and an even number of **F**s in its column in the truth-table for **P**. This is because (a) two rows assign **T** to the single atomic component of **Q** and, by the result in Exercise 1.c, **Q** has the same truth-value in these two rows; and (b) two rows assign **F** to the single atomic component of **Q** and so, by the same result, **Q** has the same truth-value in these two rows.

We will now show that if **Q** and **R** each have an even number of **T**s and an even number of **F**s in their truth-table columns, then so must **P**. Let us assume the contrary, that is, we shall suppose that **P** has an odd number of **T**s and an odd number of **F**s in its truth-table column. There are then two possibilities.

a. There are 3 **T**s and 1 **F** in **P**'s truth-table column. Then in three rows of their truth-table columns, **Q** and **R** have the same truth-value, and in one row they have different truth-values. So either **Q** has one more **T** in its truth-table column than does **R**, or vice-versa. Either way, since the sum of an even number plus 1 is odd, it follows that either **Q** has an odd number of **T**s in its truth-table column or **R** has an odd number of **T**s in its truth-table column. This contradicts our inductive hypothesis, so we conclude that **P** cannot have 3 **T**s and 1 **F** in its truth-table column.

b. There are 3 **F**s and 1 **T** in **P**'s truth-table column. By reasoning similar to that just given, it is easily shown that this is impossible, given the inductive hypothesis.

Therefore **P** must have an even number of **T**s and **F**s in its truth-table column.

9. First, a binary connective whose unit set is truth-functionally complete must be such that a sentence of which it is the main connective is false whenever all its immediate components are true. Otherwise, every sentence containing only that connective would be true whenever its atomic components were. And then, for example, the negation truth-function would not be expressible using that connective. Similar reasoning shows that the main column of the characteristic truth-table must contain **T** in the last row. Otherwise, no sentence containing that connective could be truth-functionally true.

Second, the column in the characteristic truth-table must contain an odd number of **T**s and an odd number of **F**s. For otherwise, as the induction in Exercise 8 shows, any sentence containing two atomic components and only this connective would have an even number of **T**s and an even number of **F**s in its truth-table column. The disjunction truth-function, for example, would then not be expressible.

Combining these two results, it is easily verified that there are only two possible characteristic truth-tables for a binary connective whose unit set is truth-functionally complete—that for '↓' and that for '|'.

Section 6.3E

1.a. {A ⊃ B, C ⊃ D}, {A ⊃ B}, {C ⊃ D}, ∅

b. {C ∨ ~ D, ~ D ∨ C, C ∨ C}, {C ∨ ~ D, ~ D ∨ C}, {C ∨ ~ D, C ∨ C}, {~ D ∨ C, C ∨ C}, {C ∨ ~ D}, {~ D ∨ C}, {C ∨ C}, ∅

c. {(B & A) ≡ K}, ∅

d. ∅

2.a, b, d, e.

4.a. To prove that *SD** is sound, it suffices to add a clause for the new rule to the induction in the proof of Metatheorem 6.3.1.

13. If $\mathbf{Q_{k+1}}$ at position **k** + 1 is justified by ~ ≡I, then $\mathbf{Q_{k+1}}$ is a negated biconditional.

h	**P**	
j	~ **Q**	
k + 1	~ (**P** ≡ **Q**)	**h**, **j** ~ ≡I

By the inductive hypothesis, $\Gamma_{\mathbf{h}} \vDash \mathbf{P}$ and $\Gamma_{\mathbf{j}} \vDash {\sim}\,\mathbf{Q}$. Since **P** and ~ **Q** are accessible at position **k** + 1, every member of $\Gamma_{\mathbf{h}}$ is a member of $\Gamma_{\mathbf{k+1}}$, and every member of $\Gamma_{\mathbf{j}}$ is a member of $\Gamma_{\mathbf{k+1}}$. Hence, by 6.3.2, $\Gamma_{\mathbf{k+1}} \vDash \mathbf{P}$ and $\Gamma_{\mathbf{k+1}} \vDash {\sim}\,\mathbf{Q}$. But ~ (**P** ≡ **Q**) is true whenever **P** and ~ **Q** are both true. So $\Gamma_{\mathbf{k+1}} \vDash {\sim}\,(\mathbf{P} \equiv \mathbf{Q})$ as well.

c. To show that *SD** is not sound, it suffices to give an example of a derivation in *SD** of a sentence **P** from a set Γ of sentences such that **P** is *not* truth-functionally entailed by Γ. That is, we show that for some Γ and **P**,

$\Gamma \vdash$ **P** in *SD**, but $\Gamma \nvDash$ **P**. Here is an example:

1	A	Assumption
2	A ∨ B	Assumption
3	B	1, 2 C∨E

It is easily verified that {A, A ∨ B} does not truth-functionally entail 'B'.

e. Yes. In proving Metatheorem 6.3.1, we showed that each rule of *SD* is truth-preserving. It follows that if every rule of *SD** is a rule of *SD*, then every rule of *SD** is truth-preserving. Of course, as we saw in Exercise 4.c, *adding* a rule produces a system that is not sound if the rule is not truth-preserving.

5. No. In *SD* we can derive **Q** from a sentence **P** & **Q** by &E. But, if '&' had the suggested truth-table, then {**P** & **Q**} would *not* truth-functionally entail **Q**, for (by the second row of the table) **P** & **Q** would be true when **P** is true and **Q** is false. Hence it would be the case that {**P** & **Q**} $\vdash$ **Q** in *SD* but not the case that {**P** & **Q**} $\vDash$ **Q**.

6. To prove that *SD*+ is sound for sentential logic, we must show that the rules of *SD*+ that are not rules of *SD* are truth-preserving. (By Metatheorem 6.3.1, the rules of *SD* have been shown to be truth-preserving.) The three additional rules of inference in *SD*+ are Modus Tollens, Hypothetical Syllogism, and Disjunctive Syllogism. We introduced each of these rules in Chapter 5 as a *derived* rule. For example, we showed that Modus Tollens is eliminable, that anything that can be derived using this rule can be derived without it, using just the smaller set of rules in *SD*. It follows that each of these three rules is truth-preserving. For if use of one of these rules can lead from true sentences to false ones, then we can construct a derivation in *SD* (without using the derived rule) in which the sentence derived is not truth-functionally entailed by the set consisting of the undischarged assumptions. But Metatheorem 6.3.1 shows that this is impossible. Hence each of the derived rules is truth-preserving.

All that remains to be shown, in proving that *SD*+ is sound, is that the rules of replacement are also truth-preserving. We can incorporate this as a thirteenth case in the proof of the inductive step for Metatheorem 6.3.1:

13. If $\mathbf{Q}_{\mathbf{k}+1}$ at position **k** + 1 is justified by a rule of replacement, then $\mathbf{Q}_{\mathbf{k}+1}$ is derived as follows:

h	**P**	
k + 1	[**P**]($\mathbf{Q}_1$//**Q**)	**h** RR

where RR is some rule of replacement, sentence **P** at position **h** is accessible at position **k** + 1, and [**P**]($\mathbf{Q}_1$//**Q**) is a sentence that is the result of replacing a component **Q** of **P** with a component $\mathbf{Q}_1$ in accordance with one of the rules of replacement. That the sentence **Q** is truth-functionally equivalent to $\mathbf{Q}_1$, no matter what the rule of replacement is, is easily verified. So, by Exercise 1.e in Section 6.1**E**, $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ is truth-functionally equivalent to **P**. By the inductive hypothesis, $\Gamma_{\mathbf{k}} \vDash \mathbf{P}$; and since **P** at **h** is accessible at position **k** + 1, it follows that $\Gamma_{\mathbf{k}+1} \vDash \mathbf{P}$. But $[\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$ is true whenever **P** is true (since they are truth-functionally equivalent), so $\Gamma_{\mathbf{k}+1} \vDash [\mathbf{P}](\mathbf{Q}_1//\mathbf{Q})$; that is, $\Gamma_{\mathbf{k}+1} \vDash \mathbf{Q}_{\mathbf{k}+1}$.

Section 6.4E

1. Proof of 6.4.4 Assume that $\Gamma \vdash \mathbf{P}$ in *SD*. Then there is a derivation in *SD* of the following sort

1	$\mathbf{P}_1$	
.	.	
n	$\mathbf{P}_\mathbf{n}$	
.	.	
m	**P**	

(where $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_\mathbf{n}$ are members of Γ). To show that $\Gamma \cup \{\sim \mathbf{P}\}$ is inconsistent in *SD*, we need only produce a derivation of some sentence **Q** and ~ **Q** from members of $\Gamma \cup \{\sim \mathbf{P}\}$. This is easy. Start with the derivation of **P** from Γ and add ~ **P** as a new primary assumption at line **n** + 1, renumbering subsequent lines as is appropriate. As a new last line, enter ~ **P** by Reiteration. The result is a derivation of the sort

1	$\mathbf{P}_1$	
.	.	
n	$\mathbf{P}_\mathbf{n}$	
n + 1	~ **P**	
.	.	
m + 1	**P**	
m + 2	~ **P**	**n** + 1 **R**

So if $\Gamma \vdash \mathbf{P}$, then $\Gamma \cup \{\sim \mathbf{P}\}$ is inconsistent in *SD*.

Now assume that $\Gamma \cup \{\sim \mathbf{P}\}$ is inconsistent in *SD*. Then there is a derivation in *SD* of the sort

1	$\mathbf{P}_1$	
.	.	
n	$\mathbf{P}_\mathbf{n}$	
n + 1	~ **P**	
.	.	
m	**Q**	
.	.	
p	~ **Q**	

(where $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P_n}$ all members of Γ). To show that $\Gamma \vdash \mathbf{P}$, we need only produce a derivation in which the primary assumptions are members of Γ and the last line is **P**. This is easy. Start with this derivation, but make ~ **P** an auxiliary assumption rather than a primary assumption. Enter **P** as a new last line, justified by Negation Elimination. The result is a derivation of the sort

```
    1  | P1
    .  | .
    n  | Pn
       |-----
n + 1  |  | ~ P
       |  |-----
    .  |  | .
    m  |  | Q
    .  |  | .
    p  |  | ~ Q
p + 1  | P            n + 1 – p   ~ E
```

Proof of 6.4.10. Assume $\Gamma \cup \{\mathbf{P}\}$ is inconsistent in *SD*. Then there is a derivation in *SD* of the sort

```
    1  | P1
    .  |
    n  | Pn
n + 1  | P
       |-----
    .  | .
    m  | Q
    .  | .
    p  | ~ Q
```

(where $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P_n}$ are members of Γ). But then there is also a derivation of the following sort

```
    1  | P1
    .  | .
    n  | Pn
       |-----
n + 1  |  | P
       |  |-----
       |  | .
    m  |  | Q
    p  |  | ~ Q
p + 1  | ~ P          n + 1 – p   ~ I
```

This shows that if $\Gamma \cup \{\mathbf{P}\}$ is inconsistent in *SD*, then $\Gamma \vdash$ ~ **P** in *SD*.

2. If Γ is inconsistent in *SD* then, by the definition of inconsistency in *SD*, there is some sentence **P** such that both **P** and ~ **P** are derivable in *SD* from Γ. By the definition of derivability in *SD*, there is a derivation in which all of the primary assumptions are members of Γ and **P** occurs in the scope of only those assumptions, and there is a derivation in which all of the primary assumptions are members of Γ and ~ **P** occurs in the scope of only those assumptions. Because all derivations are finite in length, it follows that only a finite subset of members of Γ occurs as primary assumptions in each of these derivations, i.e., **P** is derivable from a finite subset Γ' of Γ and ~ **P** is derivable from a finite subset Γ'' of Γ. We can extend the derivation of **P** from Γ' to a derivation of **P** from $\Gamma' \cup \Gamma''$ by adding members of Γ'' that are not members of Γ' as primary assumptions in that derivation, and we can extend the derivation of ~ **P** from Γ'' to a derivation of ~ **P** from $\Gamma' \cup \Gamma''$ by adding members of Γ' that are not members of Γ'' as primary assumptions in that derivation. This establishes that both **P** and ~ **P** are derivable from the finite subset $\Gamma' \cup \Gamma''$ of Γ, and hence that there is a finite subset of Γ that is inconsistent in *SD*.

4. Since every rule of *SD* is a rule of *SD*+, every derivation in *SD* is a derivation in *SD*+. So if $\Gamma \vDash$ **P**, then $\Gamma \vdash$ **P** in *SD*, by Metatheorem 6.4.1, and therefore $\Gamma \vdash$ **P** in *SD*+. That is, *SD*+ is complete for sentential logic.

7. a. Since we already know that *SD* is complete, we need only show that wherever Reiteration is used in a derivation in *SD*, it can be eliminated in favor of some combination of the remaining rules of *SD*. This was proved in Exercise 13.c in Section 5.4E. Hence *SD** is complete as well.

8. We used the fact that Conjunction Elimination is a rule of *SD* in proving (b) for 6.4.11, where we showed that if a sentence **P** & **Q** is a member of a set Γ^* that is maximally consistent in *SD*, then both **P** and **Q** are members of Γ^*.

9. First assume that some set Γ is truth-functionally consistent. Then obviously every finite subset of Γ is truth-functionally consistent as well, for all members of a finite subset of Γ are members of Γ, hence all are true on at least one truth-value assignment.

Now assume that some set Γ is truth-functionally inconsistent. If Γ is finite, then obviously at least one finite subset of Γ (namely, Γ itself) is truth-functionally inconsistent. If Γ is infinite, then, by Lemma 6.4.3, Γ is inconsistent in *SD*, and, by 6.4.6, some finite subset Γ' of Γ is inconsistent in *SD*—that is, for some sentence **P**, $\Gamma' \vdash$ **P** and $\Gamma' \vdash$ ~ **P**. Hence, by Metatheorem 6.3.3, $\Gamma' \vDash$ **P** and $\Gamma' \vDash$ ~ **P**, so Γ' is truth-functionally inconsistent; hence not every finite subset of Γ is truth-functionally consistent.