Philosophy 220

Truth Trees 1

Consistency:

- A set of sentences is consistent if and only if there is at least one truth-value assignment on which all the members of the set are true.
- Truth-trees are based around checking for consistency of finite sets, so the following definition will be useful:
    - A finite set Γ of sentences of *SL* is truth-functionally consistent if and only if Γ has a truth-tree with at least one completed open branch.
    - We will now proceed to explain the definitions of ‘truth-tree’ and ‘completed open branch’.

Starting a truth tree

- Consider the set {(A & B), C}
- It should be intuitively clear that the set is consistent, but the truth-tree system will demonstrate this fact.
- The first step in constructing a truth tree is to place each set member on its own line, noting that it is a set member.

1. A & B SM ← 'SM' stands for 'set member'
2. C SM

- We may now follow the next set of rules to determine if the set is consistent…

Decomposition

1. A & B SM
2. C SM

- The goal of a truth-tree is to find a truth value assignment that demonstrates the consistency or inconsistency of a finite set, so we need to eliminate (the text says ‘decompose’) all connectives (other than ‘~’).
- So we must decompose the conjunction on line 1.

Decomposing a conjunction

- A conjunction is true if and only if each of its conjuncts are true, so we can decompose the conjunction on line 1 using the rule called 'Conjunction Decomposition' (&D for short).

1.	A & B√	SM (check mark indicates decomposition)
2.	C	SM
3.	A	1, &D
4.	B	1, &D

- Notice that this is very much like doing a shortened truth table that assigns a value of T to A, to B, and to C, and finds that the set {A & B, C} is consistent.

Branch

- The truth tree we have just done does not look much like a tree (you may have noticed).
- This is because it has only one **branch**.
- A branch is all of the sentences that can be reached by starting with a sentence at the bottom of a tree and tracing an upward path through the tree, ending with the first sentence listed at the top of the tree.

1.	A & B √	SM
2.	C	SM
3.	A	1, &D
4.	B	1, &D

Open/Closed Branches:

- 'Open' and 'Closed' are mutually exclusive properties that branches hold in truth trees. All branches will be either open or closed.
  - Open branch: A branch that is not closed
  - Closed branch: a branch that contains both an atomic sentence and the negation of that atomic sentence.
- Is this tree's branch closed or open?

1.	A & B √	SM
2.	C	SM
3.	A	1, &D
4.	B	1, &D

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- Is this tree's branch closed or open?

1.	A & B √	SM
2.	C	SM
3.	A	1, &D
4.	B	1, &D

OPEN

There is not an atomic sentence and its negation on this branch.

Open/Closed Trees:

- ‘Open’ and ‘Closed’ are mutually exclusive properties that whole truth-trees hold. All trees will be either open or closed.
  - Open tree: a tree whose branches are not all closed.
  - Closed tree: a tree whose branches are all closed.
- Is this tree closed or open?

1.	A & B √	SM
2.	C	SM
3.	A	1, &D
4.	B	1, &D

Open/Closed Trees:

- 'Open' and 'Closed' are mutually exclusive properties that whole truth-trees hold. All trees will be either open or closed.
  - Open tree: a tree whose branches are not all closed.
  - Closed tree: a tree whose branches are all closed.
- Is this tree closed or open?

1.	A & B √	SM
2.	C	SM
3.	A	1, &D
4.	B	1, &D

OPEN

Its one branch is open (not closed).

Completed (Open) Branch

- We will only refer to open branches as completed or not completed.
- A completed branch is an open branch such that every sentence on it is either a **literal** (an atomic sentence or its negation) or has been decomposed. (This branch is completed)

1.	A & B √	SM
2.	C	SM
3.	A	1, &D
4.	B	1, &D

This sentence has been decomposed

Each of these is a literal

Completed Trees

- A completed tree is one such that each of its branches is either closed or completed.
- Since the tree below has only one branch, and it is a completed branch, the tree below is complete.

1.	A & B √	SM
2.	C	SM
3.	A	1, &D
4.	B	1, &D

What does it all mean?

- Since closed branches contain a contradiction (an atomic sentence and its negation, e.g. 'A & ~A'), closed branches indicate an impossible outcome.
- Since open branches are free of contradictions, all and only open branches represent possible outcomes.

What does it all mean?

1.	A & B √	SM
2.	C	SM
3.	A	1, &D
4.	B	1, &D

- The above set contains only one branch and it's open, so this is a possible result.
- In assuming that 'A & B' and 'C' are both true at the same time, we end up with a possible outcome (i.e. an open branch) which tells us that it is possible for all the members of our set to be true at the same time (which means the set is consistent). All trees with a completed open branch indicate a consistent set.

What does it all mean?

1.	A & B √	SM
2.	C	SM
3.	A	1, &D
4.	B	1, &D

- Since the above tree is completed, we can say some other things about the set. The tree shows that the set is consistent on a truth value assignment:

A	B	C
T	T	T

An inconsistent set:

- Consider the intuitively inconsistent set {A & ~B, ~A}
- A truth tree for the following looks like:

An inconsistent set:

- Consider the intuitively inconsistent set {A & ~B, ~A}
- A truth tree for the following looks like:

1. A & ~B   SM
2. ~A   SM
3. ?

An inconsistent set:

- Consider the intuitively inconsistent set {A & ~B, ~A}
- A truth tree for the following looks like:

1. A & ~B√ SM
2. ~A SM
3. A 1 &D
4. ~B 1 &D

- Now what?

An inconsistent set:

1. A & ~B√ SM
2. ~A SM
3. A 1 &D
4. ~B 1 &D

- Notice that there is again only one branch.
- Is it open or closed?

An inconsistent set:

1. A & ~B√ SM
2. ~A SM
3. A 1 &D
4. ~B 1 &D

X ('X' indicates closure)

- Notice that there is again only one branch.
- This branch is CLOSED (it contains A and ~A)

An inconsistent set:

1. A & ~B√ SM
2. ~A SM
3. A 1 &D
4. ~B 1 &D

X ('X' indicates closure)

- Notice that there is again only one branch.
- This branch is CLOSED (it contains A and ~A)
- Since there is only one branch and it is closed, the tree is closed and completed.

What does it mean?

1. A & ~B√ SM
2. ~A SM
3. A 1 &D
4. ~B 1 &D

X ('X' indicates closure)

- This tree shows that there is no possible outcome that has all members of the set true at the same time. A completed closed tree indicates an inconsistent set.

Non-Branching rules:

Conjunction decomposition (&D):

M. (P & Q)√

N. P

N+1. Q

Negated Disjunction decomposition (~ vD):

M. ~(P v Q)√

N. ~P

N+1. ~Q

Negated Negation Decomposition (~ ~D)

M. (~~P)√

N. P

Branching Rules:

Disjunction decomposition (vD):

M. (P v Q)√

N. P Q

Negated Conjunction decomposition (~ &D):

M. ~(P & Q)√

N. ~P ~Q

A Branching Tree

- Consider the set {P & Q, ~(~Q v R), P v R}
- How to start the tree?

A Branching Tree

- Consider the set {P & Q, ~(~Q v R), P v R}

1. P & Q SM
2. ~(~Q v R) SM
3. P v R SM

A Branching Tree

1.	P & Q	SM
2.	~(~Q v R)	SM
3.	P v R	SM

- What next?

A Branching Tree

1.	P & Q	SM
2.	~(~Q v R)	SM
3.	P v R	SM

- What next? We must decompose the sentences.
- Strategy tip: Always decompose non-branching sentences first. It doesn't affect the result, but it saves you space and time.
- Which are non branching?

A Branching Tree

1.	P & Q	SM
2.	~(~Q v R)	SM
3.	P v R	SM

- These are non-branching so we'll decompose these first.

A Branching Tree

1.	P & Q√	SM
2.	~(~Q v R)	SM
3.	P v R	SM
4.	P	1, &D
5.	Q	1, &D

A Branching Tree

1.	P & Q√	SM
2.	~(~Q v R)√	SM
3.	P v R	SM
4.	P	1, &D
5.	Q	1, &D
6.	~~Q	2, ~ vD
7.	~R	2, ~ vD

A Branching Tree

1.	P & Q√	SM
2.	~(~Q v R)√	SM
3.	P v R	SM
4.	P	1, &D
5.	Q	1, &D
6.	~~Q√	2, ~ vD
7.	~R	2, ~ vD
8.	Q	6, ~~D

A Branching Tree

1.	P & Q√		SM
2.	~(~Q v R)√		SM
3.	P v R √		SM
4.	P		1, &D
5.	Q		1, &D
6.	~~Q√		2, ~ vD
7.	~R		2, ~ vD
8.	Q		6, ~~D
9.	~R		2, ~ vD
10.	P	R	3, vD

A Branching Tree

1.	P & Q√	SM
2.	~(~Q v R)√	SM
3.	P v R √	SM
4.	P	1, &D
5.	Q	1, &D
6.	~~Q√	2, ~ vD
7.	~R	2, ~ vD
8.	Q	6, ~~D
9.	~R	2, ~ vD
10.	P R ✕	3, vD

This branch closes due to R and ~R in the same branch.

A Branching Tree

1.	P & Q√		SM
2.	~(~Q v R)√		SM
3.	P v R √		SM
4.	P		1, &D
5.	Q		1, &D
6.	~~Q√		2, ~ vD
7.	~R		2, ~ vD
8.	Q		6, ~~D
9.	~R		2, ~ vD
10.	P	R	3, vD

This branch is open and completed

A Branching Tree

1.	P & Q√	SM
2.	~(~Q v R)√	SM
3.	P v R √	SM
4.	P	1, &D
5.	Q	1, &D
6.	~~Q√	2, ~ vD
7.	~R	2, ~ vD
8.	Q	6, ~~D
9.	~R	2, ~ vD
10.	P R	3, vD

An open completed tree indicates that our set is consistent. What truth-value assignment demonstrates its consistency?

A Branching Tree

1.	P & Q√	SM
2.	~(~Q v R)√	SM
3.	P v R √	SM
4.	P	1, &D
5.	Q	1, &D
6.	~~Q√	2, ~ vD
7.	~R	2, ~ vD
8.	Q	6, ~~D
9.	~R	2, ~ vD
10.	P R	3, vD

An open completed tree indicates that our set is consistent. What truth-value assignment demonstrates its consistency?

P	Q	R
T		

A Branching Tree

1.	P & Q√	SM
2.	~(~Q v R)√	SM
3.	P v R √	SM
4.	P	1, &D
5.	Q	1, &D
6.	~~Q√	2, ~ vD
7.	~R	2, ~ vD
8.	Q	6, ~~D
9.	~R	2, ~ vD
10.	P R	3, vD

An open completed tree indicates that our set is consistent. What truth-value assignment demonstrates its consistency?

P	Q	R
T		F

A Branching Tree

1.	P & Q√		SM
2.	~(~Q v R)√		SM
3.	P v R √		SM
4.	P		1, &D
5.	Q		1, &D
6.	~~Q√		2, ~ vD
7.	~R		2, ~ vD
8.	Q		6, ~~D
9.	~R		2, ~ vD
10.	P	R	3, vD

An open completed tree indicates that our set is consistent. What truth-value assignment demonstrates its consistency?

P	Q	R
T	T	F

A Branching Tree

1.	P & Q√	SM
2.	~(~Q v R)√	SM
3.	P v R √	SM
4.	P	1, &D
5.	Q	1, &D
6.	~~Q√	2, ~ vD
7.	~R	2, ~ vD
8.	Q	6, ~~D
9.	~R	2, ~ vD
10.	P R	3, vD

An open completed tree indicates that our set is consistent. This truth-value assignment demonstrates its consistency.

P	Q	R
T	T	F