

Philosophy 220

Truth-Functional Equivalence and
Consistency

Review

- Logical equivalency: The members of a pair of sentences [of natural language] are logically equivalent if and only if it is not [logically] possible for one of the sentences to be true while the other sentence is false.
- Logical consistency: A set of sentences is logically consistent if and only if it is [logically] possible for all the members of that set to be true [at the same time].

Truth-Functional Equivalence:

- Sentences P and Q of SL are truth-functionally equivalent if and only if there is no truth value assignment [for the components of P and Q] on which P and Q have different truth-values.
- This means that on a full truth table, the columns for any two truth-functionally equivalent sentences of SL will be identical.

Finding Equivalence on a shortened truth-table

- As with truth-functional truth and truth-functional falsity, we test for truth-functional equivalence by looking for a counterexample.
- If we assume that one of the sentences is true and the other false, then either we will or will not get a coherent truth-value assignment. If we do, then the two sentences are shown not to be truth-functionally equivalent. If we cannot get a coherent truth-value assignment assuming that one sentence is true while the other is false, then we must try it the other way before drawing any conclusions. (why?)

$\sim(B \ \& \ \sim A)$ and $(A \vee B)$

A	B	\sim	$(B \ \& \ \sim A)$		$A \vee B$

Which columns should be identical if these two sentences are truth-functionally equivalent?

$\sim(B \ \& \ \sim A)$ and $(A \vee B)$

A	B	\sim	$(B \ \& \ \sim A)$			$A \vee B$

Which columns should be identical if these two sentences are truth-functionally equivalent?

$\sim(B \ \& \ \sim A)$ and $(A \vee B)$

A	B	\sim	$(B \ \& \ \sim A)$		$A \vee B$
		T			F

So assume that one is T and the other F.

$\sim(B \ \& \ \sim A)$ and $(A \vee B)$

A	B	\sim	$(B \ \& \ \sim A)$		$A \vee B$
		T			F

Note that the only way for $A \vee B$ to be false is for both A and B to be false.

$\sim(B \ \& \ \sim A)$ and $(A \vee B)$

A	B	\sim	$(B \ \& \ \sim A)$		$A \vee B$
F	F	T			F

Note that the only way for $A \vee B$ to be false is for both A and B to be false.

$\sim(B \ \& \ \sim A)$ and $(A \vee B)$

A	B	\sim	$(B \ \& \ \sim A)$		$A \vee B$
F	F	T			F

Now we see if this truth-assignment is coherent...

$\sim(B \ \& \ \sim A)$ and $(A \vee B)$

A	B	\sim	$(B \ \& \ \sim A)$		$A \vee B$
F	F	T	F		F

It is coherent. If A and B are both false, then $\sim(B \ \& \ \sim A)$ must be true.

$F \ \& \ (J \ \vee \ H)$ and $(F \ \& \ J) \ \vee \ H$

F	H	J	F &	(J \vee H)		(F & J)	\vee H

Which columns should be identical if these two sentences are truth-functionally equivalent?

$F \ \& \ (J \ \vee \ H)$ and $(F \ \& \ J) \ \vee \ H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H

Which columns should be identical if these two sentences are truth-functionally equivalent?

$F \ \& \ (J \ \vee \ H)$ and $(F \ \& \ J) \ \vee \ H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
			T				F

Assume one is T and the other F...

$F \& (J \vee H)$ and $(F \& J) \vee H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
	F		T			F	F

If the disjunction is false, then both disjuncts must be false.

$F \ \& \ (J \ \vee \ H)$ and $(F \ \& \ J) \ \vee \ H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
T	F		T	T		F	F

If the conjunction is true then both of its conjuncts must be true.

$F \ \& \ (J \ \vee \ H)$ and $(F \ \& \ J) \ \vee \ H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
T	F	?	T	T		F	F

Is there a truth-value assignment for J that is coherent?

$F \ \& \ (J \ \vee \ H)$ and $(F \ \& \ J) \ \vee \ H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
T	F	T	T	T		F	F

FAIL!

Is there a truth-value assignment for J that is coherent?

T is not coherent because it would make (F & J) true, which would in turn make ((F & J) ∨ H) true.

$F \ \& \ (J \ \vee \ H)$ and $(F \ \& \ J) \ \vee \ H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
T	F	F	T	T		F	F

FAIL!

Is there a truth-value assignment for J that is coherent?

F is not coherent because it would make $(J \ \vee \ H)$ false, which would in turn make $(F \ \& \ (J \ \vee \ H))$ false.

$F \& (J \vee H)$ and $(F \& J) \vee H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
T	F	X	T	T		F	F

FAIL!

Does this mean that $F \& (J \vee H)$ and $(F \& J) \vee H$ are equivalent?

$F \& (J \vee H)$ and $(F \& J) \vee H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
T	F	X	T	T		F	F

FAIL!

Does this mean that $F \& (J \vee H)$ and $(F \& J) \vee H$ are equivalent?

IT DOES NOT!

We have shown that $F \& (J \vee H)$ cannot be true while $(F \& J) \vee H$ is false, but it is still possible that $F \& (J \vee H)$ can be false while $(F \& J) \vee H$ is true.

So let's check:

$F \& (J \vee H)$ and $(F \& J) \vee H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
			F				T

So assume one is F and the other T...

$F \ \& \ (J \ \vee \ H)$ and $(F \ \& \ J) \ \vee \ H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
F	T		F				T

We have many ways to proceed here, so let us assume that F is false (to make the conjunction it is in false) and that H is true (to make the disjunction that it is in true). If this turns out to be incoherent, there are several other possibilities to try.

$F \ \& \ (J \ \vee \ H)$ and $(F \ \& \ J) \ \vee \ H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
F	T	?	F				T

Now we must see if any truth value of J would yield a coherent table...

$F \& (J \vee H)$ and $(F \& J) \vee H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
F	T	T	F				T

Let's try T first.

$F \& (J \vee H)$ and $(F \& J) \vee H$

F	H	J	F &	(J ∨ H)		(F & J)	∨ H
F	T	T	F	T		F	T

This is a coherent truth-value assignment for F, H, and J that reveals that these two sentences are not truth-functionally equivalent.

The Full Truth-Table (for illustration)

F	H	J	F &	(J \vee H)		(F & J)	\vee H
T	T	T	T	T		T	T
T	T	F	T	T		F	T
T	F	T	T	T		T	T
T	F	F	F	F		F	F
F	T	T	F	T		F	T
F	T	F	F	T		F	T
F	F	T	F	T		F	F
F	F	F	F	F		F	F

We proved with our first shortened truth-table that the first sentence is never true while the second sentence is false...

The Full Truth-Table (for illustration)

F	H	J	F &	(J \vee H)		(F & J)	\vee H
T	T	T	T	T		T	T
T	T	F	T	T		F	T
T	F	T	T	T		T	T
T	F	F	F	F		F	F
F	T	T	F	T		F	T
F	T	F	F	T		F	T
F	F	T	F	T		F	F
F	F	F	F	F		F	F

...however, we showed with the second shortened truth-table that the two are not equivalent because the second sentence can be true while the first sentence is false.

Truth-Functional Consistency

- A set of sentences of SL is truth-functionally consistent if and only if there is at least one truth value assignment [of the constituents of the set of sentences] on which all the members of the set are true.
- That means that if each of the set of sentences of SL were done on a truth-table, there would be one row of the truth table on which all of the sentences of the set are true.

Shortened tables

- Since a single example of a case in which all of the sentences of a set can be true shows that the set is consistent, when we check for consistency with a shortened truth-table, we should assume that all of the sentences of the set are true. If we get a coherent truth-value assignment from this assumption, then the set is consistent. If we cannot, then the set is inconsistent.

A consistent set?

H	J	K	$H \supset J$		$J \supset K$		$K \supset$	$\sim H$
			T		T		T	

Assume that all are true.

A consistent set?

H	J	K	$H \supset J$		$J \supset K$		$K \supset$	$\sim H$
T			T		T		T	

$H \supset J$ being true is consistent with several outcomes, so let's assume that H is true to start with.

A consistent set?

H	J	K	$H \supset J$		$J \supset K$		$K \supset$	$\sim H$
T	T		T		T		T	

If H is true, then J must be true to preserve the truth of $H \supset J$.

A consistent set?

H	J	K	$H \supset J$		$J \supset K$		$K \supset$	$\sim H$
T	T	T	T		T		T	

If J is true, then K must be true to preserve the truth of $J \supset K$.

A consistent set?

H	J	K	$H \supset J$		$J \supset K$		$K \supset$	$\sim H$
T	T	T	T		T		T	T

If K is true, then $\sim H$ must be true to preserve the truth of $K \supset \sim H$.

A consistent set?

H	J	K	$H \supset J$		$J \supset K$		$K \supset$	$\sim H$
T	T	T	T		T		T	F

FAIL!

Failure, H and $\sim H$ cannot be true at the same time.

A consistent set?

H	J	K	$H \supset J$		$J \supset K$		$K \supset$	$\sim H$
F			T		T		T	

Assume that all are true.

Then, since assuming H was true brought us an inconsistent set, let's assume H is false instead.

A consistent set?

H	J	K	$H \supset J$		$J \supset K$		$K \supset$	$\sim H$
F			T		T		T	T

If H is false, then $\sim H$ is true.

A consistent set?

H	J	K	$H \supset J$		$J \supset K$		$K \supset$	$\sim H$
F	F	F	T		T		T	T

The truth-value of $H \supset J$ is assured by H having a truth-value of false, whatever J's truth-value.

Also, the truth of $K \supset \sim H$ is assured by $\sim H$ being true, whatever K's truth-value.

So we can select any truth values for J and K so long as they don't make $J \supset K$ false. (F for both will do)

A consistent set?

H	J	K	$H \supset J$		$J \supset K$		$K \supset$	$\sim H$
F	F	F	T		T		T	T

This is one of several examples of truth-value assignments on which all three sentences end up true, so we have proven that these sentences are consistent.